Polynomial Approximation of Holomorphic Functions

Candidate Number: 760071
Degree Course: MMath Mathematics
CCD Mathematics Dissertation

Hilary Term 2017

Contents

1 Purpose

Our primary aim is to provide a concise account of Mergelyan's Theorem: setting it in context, and providing both complex analytic and functional analytic proofs which work from the fundamentals; no such account yet exists in the literature.

We begin by presenting the complex analytic proof, which is rooted in classical analysis, primarily following the structure of [21, Chapter 20]. But Mergelyan's Theorem can also be viewed as a problem in functional analysis, in particular on function algebras, and it is here that the problem finds a natural home. A proof is given in [11], but this account relies on a corpus of abstract theory and ancillary results. We present the shortest possible trip through the theory in order to prove Mergelyan's Theorem, drawing in particular on [8].

Our main sources are often terse, and written at a high level for one already familiar with the theory; our significant contribution is to produce proofs – including several entire lemmas – that fill in the (many) details left unproved and unreferenced. We also cover the necessary background material. Our goal is to produce an account which may be appreciated by the advanced undergraduate with a strong background in complex analysis, functional analysis, and abstract measure theory.

Along the way we shall encounter and prove many important results, such as the Tietze Extension Theorem, Runge's Theorem, the Lebesgue Decomposition Theorem, the Walsh–Lebesgue Theorem, and an abstract form of the F. and M. Riesz Theorem. In the final section we provide a brief survey of some further results.

It is the author's sincere hope that by the end, the reader will share our fascination with this topic.

2 Preliminaries

2.1 Context

The problem of approximating certain classes of functions by those of simpler classes is the concern of 'approximation theory', which in general has many different aspects. With what functions can we approximate what functions? Over which sets? In what fashion do they converge, and how fast? What is the optimal approximant subject to certain constraints? (For example, that an approximating polynomial must be of order at most n.) More esoteric concerns exist too, see [22, Section 0]. We focus on the first two questions – in particular, the problem of uniformly approximating by polynomials on compact subsets of the complex plane.

Which functions might we hope to approximate? Polynomials are continuous; their uniform limit will be also. So at the very least we must restrict attention to continuous functions. Indeed, in 1885, Weierstrass proved one of the first results of this form, the now-classical Weierstrass Approximation Theorem. This may be generalised, for example by the Stone–Weierstrass Theorem, but we might also seek to generalise in other ways: in particular, to generalise the domain under consideration; the natural generalisation is to compact subsets of the complex plane. But generalising so broadly does not come for free: polynomials are holomorphic functions; their uniform limit will be also. Thus it is necessary to restrict attention to those functions which are holomorphic in the interior of the domain, a condition which was trivially satisfied in the Weierstrass Approximation Theorem – when the domain, viewed as a subset of the compex plane, lacked an interior.

One other condition is also needed, which is that the complement of the domain must be connected. The necessity of this condition is a little less obvious. If f is uniformly approximated on the domain by polynomials (P_n), then the maximum principle implies that (P_n) converges uniformly (to a holomorphic function) on all bounded connected components of the complement of the domain: this then defines a holomorphic extension of f. Hence in particular, any function lacking such a holomorphic extension (for example, it might have a pole in one of the bounded connected components of the complement) cannot be approximated by polynomials. Thus we reduce to the case of connected complement.

Whilst all of these conditions are necessary, it is not at all obvious that they are sufficient. Mergelyan's Theorem asserts that they are:

Mergelyan's Theorem. *Let $K \subseteq \mathbb{C}$ be compact, and such that $\mathbb{C} \setminus K$ is connected. Then f is uniformly approximable on K by polynomials in z if and only if f is continuous on K and holomorphic in the interior of K.*

This was first proved by Mergelyan [20, Theorem 1.4] in 1951 (published in 1952), and offers a complete solution to the problem of polynomial approximation in the complex plane.

There is an important precursor to Mergelyan's Theorem, called Runge's Theorem, which gives sufficient but not necessary conditions for polynomial approximation on the domain K: that the function to be approximated should be holomorphic on some slightly larger region Ω, open and containing K. It is to Runge's Theorem that the concluding step of Mergelyan's Theorem appeals.

Historically, a great many theorems have stood between those of Weierstrass and Mergelyan. Walsh generalised the Weierstrass approximation theorem to domains which are arbitrary Jordan curves not separating the plane, which was further generalised in 1930 by Hartogs and Rosenthal, who removed the topological restriction. It was in 1934 that Lavrent'ev gave the complete solution to polynomial approximation of continuous functions, by showing that it be necessary and sufficient that the domain be nowhere dense and not separate the plane, see [11, Chapter II, Theorem 8.7]. See [20, Section 1] for historical references.

The question still remained open, however, regarding those domains that have nonempty interior. Conditions on the smoothness of the boundary proved important, tying in to the theory of conformal maps. For instance, in 1926, Walsh proved an approximation result for those regions whose boundary was homeomorphic to a circle. In 1945, Keldysh proved that it was sufficient that the domain be the closure of some bounded region on which the function of interest is analytic. Many more exotic conditions were also shown to be sufficient, again [20, Section 1] provides historical references. Mergelyan's Theorem supersedes them all.

The condition that the complement of the domain should be connected remains unsatisfying, and it is here that the theory continues to develop; Mergelyan's Theorem only scratches the surface. Consider now the case when the complement of the domain is not connected. Our above discussion shows that it is certainly insufficient to consider approximation by polynomials. Instead, it turns out that the appropriate objects of study are *rational functions*, that is, a quotient of two polynomials. Many of the polynomial approximation results may be generalised to rational functions: for example, the version of Runge's Theorem that we present here is actually a restricted version of the usual 'Runge's Theorem', which allows for domains whose complement is disconnected. However, not every result can be generalised as completely as we may hope – in particular, Mergelyan's Theorem does not generalise to arbitrary compact sets, and in general, the question of rational approximation still remains open. We provide a brief exposition of some results on rational approximation at the end of this paper; see Section 5.

Of course, we may continue to generalise our problem in other ways: why should we restrict ourselves to domains in $\mathbb{C}$? We might reasonably consider domains in $\mathbb{C}^n$ (or on Riemann surfaces). Here we begin to enter the realm of the theory of several complex variables, for which even the question of polynomial approximation remains open. See perhaps [26, Section 8].

2.2 Notations and Definitions

For completeness' sake we list some of the standard notation that we use.

(i) The extended real line is denoted $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. The Riemann sphere is denoted $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

(ii) Let $E \subseteq \widehat{\mathbb{C}}$. Then E° denotes the interior of E.

(iii) Let $E \subseteq \mathbb{C}$ be nonempty, $z \in \mathbb{C}$, and $f : E \to \mathbb{C}$. Then $\operatorname{dist}(z, E)$ denotes the distance from z to E, $\operatorname{diam}(E)$ denotes the diameter of E, and $\operatorname{supp}(f)$ denotes the support of f.

(iv) Let $\gamma\colon [0,1] \to \mathbb{C}$ be a simple closed path. Then $\mathrm{int}(\gamma)$ denotes the interior of γ.

(v) Let $E \subseteq \widehat{\mathbb{C}}$ be open. Then $H(E)$ denotes the set of all functions $f\colon \widehat{\mathbb{C}} \to \mathbb{C}$ which are holomorphic in E.

(vi) Let $E \subseteq \widehat{\mathbb{C}}$. Then $C_{\mathbb{R}}(E)$, $C(E)$ denote the spaces of all real valued, respectively complex valued, continuous functions $f\colon E \to \mathbb{C}$, equipped with the supremum norm. Further, $C_c(E)$ denotes the subspace of $C(E)$ consisting of those functions whose support is compact, and $C^k(E)$ denotes the subspace of $C(E)$ consisting of those functions which are k-times continuously differentiable. (They do not necessarily have complex derivatives.) These have intersection $C_c^k(E) = C_c(E) \cap C^k(E)$. Finally, $A(E)$ is the subspace of $C(E)$ consisting of its intersection with $H(E^\circ)$.

(vii) Let $z \in \mathbb{C}$, and $\delta > 0$. Then $B(z,\delta)$ denotes the open disc in $\mathbb{C}$ centred on z, with radius δ. Further, $B'(z,\delta)$ will denote the punctured disc, and $\overline{B}(z,\delta)$ will denote the closed disc.

(viii) The Laplace operator is denoted $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Definition 2.2.1. Let $K \subseteq \mathbb{C}$ be compact. Then let $P(K)$ denote the set of all functions $f\colon K \to \mathbb{C}$ which are uniform limits on K of polynomials in z.

Definition 2.2.2. The *Wirtinger derivative* $\partial_{\bar{z}}$ is defined by

$$\partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right) = \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}\left(\frac{\partial}{\partial r} + \frac{\mathrm{i}}{r}\frac{\partial}{\partial \theta}\right).$$

For f a holomorphic function, the Cauchy–Riemann equations imply that $\partial_{\bar{z}} f = 0$.

2.3 Initial Results

We begin by proving a few miscellaneous results which will be of use to both the complex analytic and the functional analytic approaches. These results are mostly elementary, we follow [21, Theorem 20.3] and [15]. The proof of the Tietze Extension Theorem is standard, we follow [21, Theorem 20.4].

Lemma 2.3.1. *Let $\Phi \in C_c^1(\mathbb{C})$. Then the following formula holds, with $w = \zeta + \mathrm{i}\eta$:*

$$\Phi(z) = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{(\partial_{\bar{z}}\Phi)(w)}{w - z}\,\mathrm{d}\zeta\,\mathrm{d}\eta.$$

Proof. Fix z and let $w = z + r\mathrm{e}^{\mathrm{i}\theta}$. Then using the polar description of $\partial_{\bar{z}}$,

$$\begin{aligned}
-\frac{1}{\pi}\iint_{\mathbb{C}} \frac{(\partial_{\bar{z}}\Phi)(w)}{w-z}\,\mathrm{d}\zeta\,\mathrm{d}\eta &= -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{(\partial_{\bar{z}}\Phi)(w)}{r\mathrm{e}^{\mathrm{i}\theta}}\cdot r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= -\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}(\frac{\partial}{\partial r}+\frac{\mathrm{i}}{r}\frac{\partial}{\partial\theta})(\Phi)(w)\,\mathrm{d}r\,\mathrm{d}\theta \\
&= -\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}\frac{\partial\Phi}{\partial r}(w)\,\mathrm{d}r\,\mathrm{d}\theta \\
&= -\frac{1}{2\pi}\int_0^{2\pi}\left[\Phi(z+r\mathrm{e}^{\mathrm{i}\theta})\right]_0^{\infty}\,\mathrm{d}\theta \\
&= \Phi(z).
\end{aligned}$$

The third line uses that Φ is 2π periodic in θ to conclude that the integral of $\frac{\partial\Phi}{\partial\theta}$ is zero, and the fifth line uses compact support to get that Φ is zero at infinity. $\square$

Lemma 2.3.2. *Let $K \subseteq \mathbb{C}$ be compact. Suppose that $\lambda \in \mathbb{C}$ and $f, g \in P(K)$. Then $f+g, fg, \lambda f \in P(K)$. Suppose instead that $h_n \in P(K)$ for all $n \in \mathbb{N}$, and that $h_n \to h$ as $n \to \infty$, uniformly on K. Then $h \in P(K)$.*

This is trivial to show; the proof is omitted. The lemma is essentially stating that $P(K)$ is a closed subalgebra of $C(K)$. (In fact it is a *commutative Banach algebra*, see [21, Chapter 18], or [26].)

Lemma 2.3.3 (Pole-pushing lemma)**.** *Let $K \subseteq \mathbb{C}$ be compact, and such that $\mathbb{C}\setminus K$ is connected. For $\alpha \in \mathbb{C}$, let $h_\alpha(z) = 1/(\alpha - z)$. Then $\alpha \notin K$ implies that $h_\alpha \in P(K)$.*

Proof. Let $X = \{\alpha \in \mathbb{C}\setminus K : h_\alpha \in P(K)\}$. We will show that $X = \mathbb{C}\setminus K$.

For α such that $|\alpha| > |z|$, expand $h_\alpha(z)$ by the geometric series formula to give that

$$h_\alpha(z) = \frac{1}{\alpha - z} = \frac{1}{\alpha}(1 + \frac{z}{\alpha} + \frac{z^2}{\alpha^2} + \cdots).$$

This sum is uniformly convergent in $\overline{B}(0, \frac{1}{2}|\alpha|)$, say. As K is bounded, α may be chosen sufficiently large so that $K \subseteq \overline{B}(0, \frac{1}{2}|\alpha|)$, and hence that $\alpha \in X$ for such α. So in particular, X is nonempty.

Now let $\alpha \in X$ and suppose $\beta \in \mathbb{C}\setminus K$ is such that $|\alpha - \beta| < \operatorname{dist}(\alpha, K)$. As $h_\alpha \in P(K)$, repeated applications of Lemma 2.3.2 give that g_N defined by

$$g_N(z) = 1 + \frac{\alpha-\beta}{\alpha-z} + \frac{(\alpha-\beta)^2}{(\alpha-z)^2} + \cdots + \frac{(\alpha-\beta)^N}{(\alpha-z)^N}$$

is such that $g_N \in P(K)$. Now again expand according to the geometric series formula, to give that

$$h_\beta(z) = \frac{1}{(\alpha-z)(1-\frac{\alpha-\beta}{\alpha-z})} = \frac{1}{\alpha-z}(1+\frac{\alpha-\beta}{\alpha-z}+\frac{(\alpha-\beta)^2}{(\alpha-z)^2}+\cdots) = \frac{1}{\alpha-z}\lim_{N\to\infty} g_N(z).$$

By choice of β, this sum is uniformly convergent in K. So Lemma 2.3.2 gives that $h_\beta \in P(K)$ also, and so $\beta \in X$. Thus we have shown that $\beta \in \mathbb{C}\setminus K$, $\alpha \in X$ and $|\alpha - \beta| < \operatorname{dist}(\alpha, K)$ together imply that $\beta \in X$.

Now pick any $\beta \in \mathbb{C} \setminus K$. Since $\mathbb{C} \setminus K$ is open and connected, it is path connected, so there exists some path γ from α to β in $\mathbb{C} \setminus K$. The image of γ is compact and disjoint from K, hence there is some positive distance between them. Let $\delta > 0$ be half this distance. As γ is uniformly continuous, there exists some finite set of points t_j such that $0 = t_0 < t_1 < \cdots < t_m = 1$, and such that $|\gamma(t_j) - \gamma(t_{j+1})| < \delta$ for all $j < m$. Now $\gamma(t_0) = \alpha \in X$, and $|\gamma(t_0) - \gamma(t_1)| < \delta < \operatorname{dist}(\alpha, K)$, so by what we have just shown, $\gamma(t_1) \in X$ also. Repeat to get that $\gamma(t_j) \in X$ for all j, and hence that in particular $\beta = \gamma(t_m) \in X$. Hence $X = \mathbb{C} \setminus K$. $\square$

Remark 2.3.4. It follows from this result that all rational functions with poles off K are in $P(K)$. This fact will prove useful later, in Section 5. We have essentially 'pushed' their poles to ∞. More general pole-pushing results exist, see [16, Lemma 12.1.6].

Lemma 2.3.5 (Urysohn's Lemma). *Let $K \subseteq \mathbb{C}$ be compact. Let $U \subseteq \mathbb{C}$ be open and contain K. Then there exists $f \in C_{\mathbb{R}}(\mathbb{C})$, with support in $\overline{U}$, such that $\mathbb{1}_K(z) \leqslant f(z) \leqslant \mathbb{1}_U(z)$ for all $z \in \mathbb{C}$.*

Proof. Define f by

$$f(z) = \frac{\operatorname{dist}(z, \mathbb{C} \setminus U)}{\operatorname{dist}(z, K) + \operatorname{dist}(z, \mathbb{C} \setminus U)}.$$

It is clear that this has the required properties. $\square$

Theorem 2.3.6 (Tietze Extension Theorem, Real Case). *Let $K \subseteq \mathbb{C}$ be compact, and let $f \in C_{\mathbb{R}}(K)$. Then there exists $F \in C_c(\mathbb{C})$, real valued, such that $F|_K = f$. Furthermore $\sup_{z\in\mathbb{C}}|F(z)| = \sup_{z\in K}|f(z)|$, and $\inf_{z\in\mathbb{C}}|F(z)| = \inf_{z\in K}|f(z)|$.*

Proof. As its domain is compact, f must be bounded. By rescaling, assume without loss of generality that $\inf_{z\in K}|f(z)| = -1$ and $\sup_{z\in K}|f(z)| = 1$. Let $\Omega \subseteq \mathbb{C}$ be open, bounded, and contain K. Let

$$K^+ = \{z \in K : f(z) \geqslant \frac{1}{3}\},$$
$$K^- = \{z \in K : f(z) \leqslant -\frac{1}{3}\}.$$

Then these are disjoint compact subsets of Ω, so as a consequence of Urysohn's Lemma, there exists $f_1 \in C(\mathbb{C})$, with support in $\overline{\Omega}$, such that $f_1(z) = \pm\frac{1}{3}$ for $z \in K^{\pm}$, and such that $-\frac{1}{3} \leqslant f(z) \leqslant \frac{1}{3}$ for all $z \in \Omega$. Thus for $z \in K$ and $w \in \Omega$,

$$|f(z) - f_1(z)| \leqslant \frac{2}{3},$$
$$|f_1(w)| \leqslant \frac{1}{3}.$$

This construction may be repeated with $f - f_1$ in place of f, to similarly generate an f_2. Inductively repeat this to produce a sequence (f_n), such that for $z \in K$ and $w \in \Omega$,

$$\left| f(z) - \sum_{i=1}^{n} f_i(z) \right| \leqslant \left(\frac{2}{3}\right)^n, \tag{1}$$

$$|f_n(w)| \leqslant \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1}. \tag{2}$$

Then F may be defined to be $\sum_{i=1}^{\infty} f_i$ on Ω, and zero in $\mathbb{C} \setminus \Omega$. By (2) this sum converges uniformly on Ω, so F is continuous. Each f_n has support in $\overline{\Omega}$, hence F has support in $\overline{\Omega}$. And (1) guarantees that F converges to f in K. Finally, (2) also implies that $\sup_{z\in\mathbb{C}}|F(z)| = 1 = \sup_{z\in K}|f(z)|$ and $\inf_{z\in\mathbb{C}}|F(z)| = -1 = \inf_{z\in K}|f(z)|$. $\square$

Theorem 2.3.7 (Tietze Extension Theorem)**.** *Let $K \subseteq \mathbb{C}$ be compact, and let $f \in C(K)$. Then there exists $F \in C_c(\mathbb{C})$ such that $F|_K = f$.*

Proof. Apply the real case of the Tietze Extension Theorem to $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ to find $u, v \in C_c(\mathbb{C})$ real valued such that $u|_K = \mathrm{Re}(f)$ and $v|_K = \mathrm{Im}(f)$. Then it is clear that F defined by $F = u + iv$ has the necessary properties. $\square$

Remark 2.3.8. In the context of metric spaces, Urysohn's Lemma is essentially a triviality – we state it separately only so that we may use it later. However, both Urysohn's Lemma and the Tietze Extension Theorem are in fact topological results, holding on locally compact Hausdorff spaces. In this case, Urysohn's Lemma becomes somewhat more difficult to prove. The proof of the Tietze Extension Theorem carries through without any changes. It is possible to avoid Urysohn's Lemma and find a metric-spaces-only proof of the Tietze Extension Theorem, see [10, Exercise 4.1.F], which references [18]. This uses a construction due to Hausdorff, but the proof is somewhat fiddly.

3 Complex Analytic Proof

It was a classical complex analytic proof that was first provided by Mergelyan [20], which has since been refined into the version we provide here. We primarily use [21, Chapter 20], see also [16, Theorem 12.2.1] for an alternative presentation.

3.1 Complex Analytic Preliminaries

We begin by stating a few definitions and standard results, without proof. Consult [21, Theorem 14.14] or [16, Theorem 13.1.6] for a proof of the Koebe $\frac{1}{4}$ Theorem; consult [21, Theorem 14.8] or [16, Section 6.7] for a proof of the Riemann Mapping Theorem.

Definition 3.1.1. Let $z \in \mathbb{C}$ and $r > 0$. Then $\Gamma(z, r)$ is the circle with centre z and radius r, oriented anticlockwise.

Definition 3.1.2. A *curve* shall be taken to mean a continuous, piecewise continuously differentiable function from $[0, 1]$ to $\mathbb{C}$, or the image of such a function.

Definition 3.1.3. Let $E \subseteq \mathbb{C}$. Define the *modulus of continuity* ω of some function $f\colon E \to \mathbb{C}$ to be

$$\omega(\delta) = \sup\{|f(z_1) - f(z_2)| \,:\, z_1, z_2 \in E, |z_1 - z_2| < \delta\}.$$

Note that $\omega(\delta) \to 0$ as $\delta \to 0$ if and only if f is uniformly continuous on E.

Theorem 3.1.4 (Koebe $\frac{1}{4}$ Theorem)**.** *Suppose $f \in H(B(0,1))$ is injective and such that $f(0) = 0$ and $f'(0) = 1$. Then $B(0, \frac{1}{4}) \subseteq f(B(0,1))$.*

Theorem 3.1.5 (Riemann Mapping Theorem)**.** *Let $\Omega \subseteq \widehat{\mathbb{C}}$ be open, simply connected, and such that $\widehat{\mathbb{C}} \setminus \Omega$ contains at least two distinct points. Then there exists a conformal (biholomorphic) map from $B(0,1)$ to Ω.*

3.2 Runge's Theorem

We begin by proving Runge's Theorem (also occasionally known as the Runge–Walsh Theorem) following [15], see also [16, Proposition 12.1.5]. Runge's Theorem may be viewed as the motivation behind Mergelyan's Theorem: it gives sufficient conditions for uniform approximation by polynomials, in particular that our function be holomorphic on a slightly larger set. Mergelyan's Theorem then shows that sufficient and necessary conditions are actually slightly weaker than that.

Lemma 3.2.1. *Let $K \subseteq \mathbb{C}$ be compact, and let $F\colon K \times [0,1] \to \mathbb{C}$ be continuous. Then $G(z)$ defined by*

$$G(z) = \int_0^1 F(z,t)\,\mathrm{d}t$$

may be uniformly approximated on K as $N \to \infty$ by $G_N(z)$ defined by

$$G_N(z) = \frac{1}{N}\sum_{k=0}^{N-1} F(z, \frac{k}{N}).$$

Proof. It is clear from the definition of G and G_N that

$$|G(z) - G_N(z)| \leqslant \sup\{|F(z,x) - F(z,y)| \,:\, x, y \in [0,1], |x-y| \leqslant \frac{1}{N}\}.$$

Furthermore, as $K \times [0,1]$ is compact, F is uniformly continuous; let F have modulus of continuity ω. Then

$$\begin{aligned}\sup_{z\in K}|G(z) - G_N(z)| &\leqslant \sup\{|F(z,x) - F(z,y)| \,:\, x, y \in [0,1], |x-y| \leqslant \frac{1}{N}, z \in K\}\\ &\leqslant \omega(\frac{1}{N}).\end{aligned}$$

This tends to zero as $N \to \infty$. Hence G_N uniformly approximates G. □

Theorem 3.2.2 (Runge's Theorem)**.** *Let $K \subseteq \mathbb{C}$ be compact, and such that $\mathbb{C} \setminus K$ is connected. Suppose that Ω is an open set containing K, such that $f \in H(\Omega)$. Then $f \in P(K)$.*

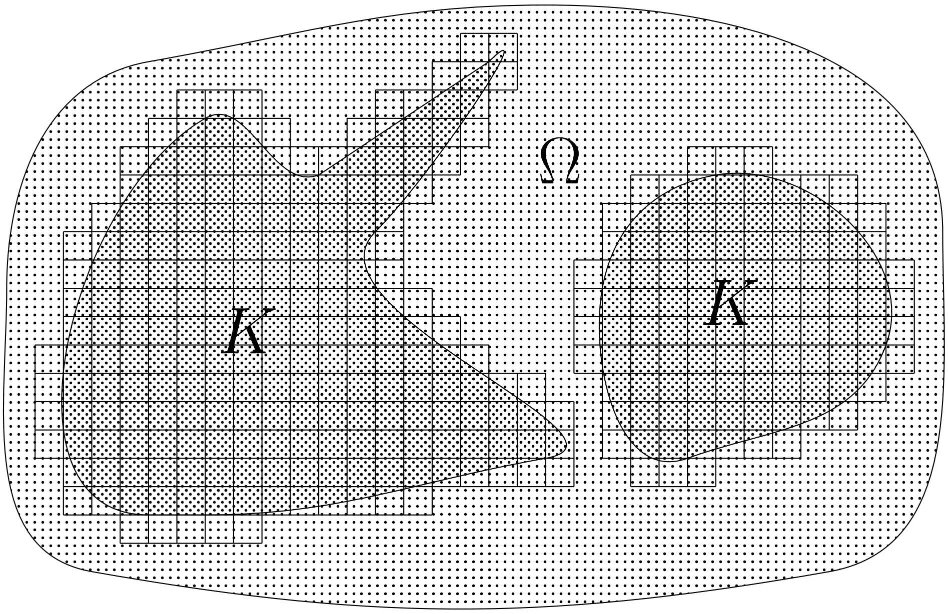

Figure 1: Runge's Theorem square contours (square side length not to scale)

Proof. As K is compact and $\mathbb{C} \setminus \Omega$ is closed, $\delta = \inf_{z \in K} \operatorname{dist}(z, \mathbb{C} \setminus \Omega)$ is such that $\delta > 0$. Now take a grid of squares of side length $\delta/10$, say. Let $\gamma_1, \ldots, \gamma_n$ be the boundaries of those squares that intersect K, treated as contours oriented anticlockwise. There are a finite number of these, as K is compact. We include those squares that intersect K only at their boundary. So all γ_j are in Ω. See Figure 1 for an example where K is disconnected.

Now fix $z \in K$. Suppose z lies in the interior of a square; so $z \in \operatorname{int}(\gamma_i)$ for one particular i. Then by Cauchy's Formula,

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma_i} \frac{f(w)}{w - z} \, \mathrm{d}w.$$

And for $j \neq i$, by Cauchy's Theorem,

$$0 = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma_j} \frac{f(w)}{w - z} \, \mathrm{d}w$$

as $f(w)/(w - z)$ will be holomorphic there. Hence for all such z,

$$f(z) = \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{n} \oint_{\gamma_j} \frac{f(w)}{w - z} \, \mathrm{d}w.$$

This statement is also true for those $z \in K$ that lie on the edges of our squares, by applying Cauchy's Formula to the rectangular contour formed by taking the union of the square contours either side, and removing their common edge (on which z lies). Similarly, the statement is true for those $z \in K$ that lie on the boundary of four squares, at their common corner.

So now remove every edge that is shared by two γ_j to form a collection of edges $E_1, \ldots, E_m$, disjoint from K. So for all $z \in K$,

$$\begin{aligned} f(z) &= \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{m} \int_{E_j} \frac{f(w)}{w-z} \,\mathrm{d}w \\ &= \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{m} F_j(z), \end{aligned} \tag{3}$$

where $F_j \colon K \to \mathbb{C}$ is defined by

$$F_j(z) = \int_0^1 G_j(z,t)\,\mathrm{d}t$$

and $G_j \colon K \times [0,1] \to \mathbb{C}$ is defined by

$$G_j(z,t) = \frac{f(E_j(t))}{E_j(t) - z} E_j'(t).$$

Now the G_j are continuous, so by Lemma 3.2.1, each F_j may be uniformly approximated (as $N \to \infty$) by $F_{j,N}$ defined by

$$\begin{aligned} F_{j,N}(z) &= \frac{1}{N} \sum_{k=0}^{N-1} G(z, \frac{k}{N}) \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \frac{f(E_j(\frac{k}{N}))}{E_j(\frac{k}{N}) - z} E_j'(\frac{k}{N}) \\ &= \frac{1}{N} \sum_{k=0}^{N-1} f(E_j(\frac{k}{N})) E_j'(\frac{k}{N}) h_{E_j(\frac{k}{N})}(z), \end{aligned}$$

where we recall that $h_\alpha(z) = 1/(\alpha - z)$, as defined in the pole-pushing lemma (Lemma 2.3.3).

Now each E_j is disjoint from K, so $E_j(\frac{k}{N}) \notin K$ for all $k \in \{0, \ldots, N-1\}$. Hence by the pole-pushing lemma (Lemma 2.3.3), $h_{E_j(k/N)} \in P(K)$.

Thus by repeated application of Lemma 2.3.2, $F_{j,N} \in P(K)$ also. Apply the lemma again, using the fact that $F_{j,N} \to F_j$ uniformly as $N \to \infty$, to get that $F_j \in P(K)$ also. A final application of the same lemma to (3) then gives that $f \in P(K)$ as well. $\square$

3.3 Mergelyan's Theorem

Now that we have the Tietze Extension Theorem and Runge's Theorem in hand, we are ready to move on to tackling Mergelyan's Theorem. Proving this requires some perseverance. We begin by extending our function of interest f to the whole complex plane, via the Tietze Extension Theorem. We then smooth it out by convolving with a mollifier, to construct a function Φ that approximates f, and is 'almost holomorphic' on the whole plane. This is then used to construct a function F that is holomorphic on a slightly larger set, and approximates Φ. Finally it is to F that we apply Runge's Theorem.

Thus it is a consequence of the proof of Mergelyan's Theorem that it is possible to uniformly approximate functions holomorphic on a compact set by functions which are holomorphic on a slightly larger set, also see [11, Chapter VIII, Theorem 7.4].

Lemma 3.3.1. *Fix $\delta > 0$ and let $s\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be defined by*

$$s(r) = \begin{cases} \dfrac{3}{\pi\delta^2}(1 - \dfrac{r^2}{\delta^2})^2 & \text{for } 0 \leqslant r \leqslant \delta, \\ 0 & \text{for } r > \delta. \end{cases}$$

Also define $S\colon \mathbb{C} \to \mathbb{R}_{\geqslant 0}$ by $S(z) = s(|z|)$.

Then $S \in C_c^1(\mathbb{C})$, and satisfies the following properties, with $z = x + \mathrm{i}y$:

$$S(z) = 0 \quad \text{for } |z| \geqslant \delta, \tag{4}$$

$$\iint_{\mathbb{C}} S(z)\,\mathrm{d}x\,\mathrm{d}y = 1, \tag{5}$$

$$\iint_{\mathbb{C}} \partial_{\bar{z}} S(z)\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{6}$$

$$\iint_{\mathbb{C}} |\partial_{\bar{z}} S(z)|\,\mathrm{d}x\,\mathrm{d}y = \frac{24}{15\delta} < \frac{2}{\delta}. \tag{7}$$

Proof. By substituting $r^2 = x^2 + y^2$ into the definition of s, it is clear that S has continuous partial derivatives in x and y.[1] Hence $S \in C_c^1(\mathbb{C})$ as we also have compact support. Furthermore (4) is immediate from the definition.

We simply compute (5), with $z = r\mathrm{e}^{\mathrm{i}\theta}$:

$$\begin{aligned}\iint_{\mathbb{C}} S(z)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{2\pi}\int_0^{\infty} S(r\mathrm{e}^{\mathrm{i}\theta}) r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 2\pi \int_0^{\delta} s(r) r\,\mathrm{d}r \\ &= \int_0^{\delta} \frac{6}{\delta^2} r(1 - \frac{r^2}{\delta^2})^2\,\mathrm{d}r \\ &= 1.\end{aligned}$$

Next, (6) is immediate as S has compact support. Explicitly, using the fact that S is independent of θ,

$$\begin{aligned}\iint_{\mathbb{C}} \partial_{\bar{z}} S(z)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{2\pi}\int_0^{\infty} \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}(\frac{\partial}{\partial r} + \frac{\mathrm{i}}{r}\frac{\partial}{\partial\theta}) S(z) r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \frac{1}{2}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta \int_0^{\infty} \frac{\partial S}{\partial r} r\,\mathrm{d}r \\ &= 0.\end{aligned}$$

[1]Note how there is no issue at either the origin or at δ.

Lastly (7) follows by similar computation:

$$
\begin{aligned}
\iint_{\mathbb{C}} |\partial_{\bar{z}} S(z)| \,\mathrm{d}x\,\mathrm{d}y &= \int_0^{2\pi} \int_0^{\infty} \left| \frac{1}{2} \mathrm{e}^{\mathrm{i}\theta} (\frac{\partial}{\partial r} + \frac{\mathrm{i}}{r} \frac{\partial}{\partial \theta}) S(z) \right| r \,\mathrm{d}r \,\mathrm{d}\theta \\
&= \frac{1}{2} \int_0^{2\pi} \mathrm{d}\theta \int_0^{\infty} \left| \frac{\partial S}{\partial r} \right| r \,\mathrm{d}r \\
&= -\pi \int_0^{\delta} s'(r) r \,\mathrm{d}r \\
&= \frac{12}{\delta^4} \int_0^{\delta} (r^2 - \frac{r^4}{\delta^2}) \,\mathrm{d}r \\
&= \frac{24}{15\delta} \\
&< \frac{2}{\delta}.
\end{aligned}
$$

□

Lemma 3.3.2. *Suppose that $F \in H(B'(0,1))$ is injective and has a simple pole at* 0 *with residue a. Then* $\operatorname{diam}(\widehat{\mathbb{C}} \setminus F(B(0,1))) \leqslant 4|a|$.

Proof. Pick any $w_1, w_2 \in \widehat{\mathbb{C}} \setminus F(B(0,1))$. We shall show that $|w_1 - w_2| \leqslant 4|a|$. Define $g(z) = a/(F(z) - w_1)$. Certainly this is injective, and

$$g(0) = \frac{a}{F(0) - w_1} = 0.$$

By the definition of F, it will have an expansion

$$F(z) = \frac{a}{z} + \sum_{n=0}^{\infty} c_n z^n,$$

and so

$$F'(z) = -\frac{a}{z^2} + \sum_{n=1}^{\infty} n c_n z^{n-1}.$$

Hence

$$g'(z) = \frac{-aF'(z)}{(F(z) - w_1)^2} = \frac{a\left(az^{-2} - \sum_{n=1}^{\infty} n c_n z^{n-1}\right)}{\left(az^{-1} + \sum_{n=0}^{\infty} c_n z^n - w_1\right)^2} = \frac{1 - a^{-2} \sum_{n=1}^{\infty} n c_n z^{n+1}}{\left(1 + a^{-1} \sum_{n=0}^{\infty} c_n z^{n+1} - w_1 z\right)^2},$$

and so $g'(0) = 1$.

Hence by the Koebe $\frac{1}{4}$ Theorem (Theorem 3.1.4), $B(0, \frac{1}{4}) \subseteq g(B(0,1))$. Hence $\widehat{\mathbb{C}} \setminus \overline{B}(0,4) \subseteq a^{-1}(F(B(0,1)) - w_1)$. (Seen by applying $z \mapsto 1/z$ to these regions.) Thus $\widehat{\mathbb{C}} \setminus [a^{-1}(F - w_1)(B(0,1))] \subseteq \overline{B}(0,4)$, so $\widehat{\mathbb{C}} \setminus F(B(0,1)) \subseteq \overline{B}(w_1, 4|a|)$. In particular $w_2 \in \overline{B}(w_1, 4|a|)$, hence $|w_1 - w_2| \leqslant 4|a|$. □

Lemma 3.3.3. *Let $\beta \in \mathbb{C}$ and $r > 0$. Let $D = B(\beta, r)$, and let $E \subseteq D$ be compact, simply connected and of diameter at least r. Let $\Omega = \widehat{\mathbb{C}} \setminus E$. Then there exists $g \in H(\Omega)$ and $\alpha \in \mathbb{C}$ such that for all $z \in \Omega$ and all $w \in D$,*

$$|Q(z,w)| < \frac{100}{r}, \tag{8}$$

$$\left|Q(z,w) - \frac{1}{z-w}\right| < \frac{1\,000r^2}{|z-w|^3}, \tag{9}$$

where Q is given by $Q(z,w) = g(z) + (w-\alpha)(g(z))^2$.

Proof. First assume without loss of generality that $\beta = 0$: the result in general follows simply by composition with translation.

By the Riemann Mapping Theorem (Theorem 3.1.5), there is a conformal map $F \colon B(0,1) \to \Omega$. By precomposing with a conformal automorphism of the unit disc, we may assume that $F(0) = \infty$. Thus it will have a Laurent expansion of the form

$$F(z) = \frac{a}{z} + \sum_{n=0}^{\infty} c_n z^n.$$

Now define g by

$$g(z) = \frac{1}{a} F^{-1}(z),$$

so that g is a conformal map from Ω to $B(0, |a|^{-1})$. Then define α by

$$\alpha = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma(0,r)} z g(z)\,\mathrm{d}z.$$

Now we show that these satisfy (8) and (9).

Apply Lemma 3.3.2 to F to get that

$$r \leqslant \operatorname{diam}(E) = \operatorname{diam}(\widehat{\mathbb{C}} \setminus \Omega) = \operatorname{diam}(\widehat{\mathbb{C}} \setminus F(B(0,1))) \leqslant 4|a|.$$

So as g is a conformal mapping of Ω onto $B(0, |a|^{-1})$, this implies that for all $z \in \Omega$,

$$|g(z)| \leqslant |a|^{-1} < \frac{4}{r} \tag{10}$$

and hence that

$$|\alpha| \leqslant \frac{1}{2\pi} \cdot 2\pi r \cdot \sup_{z \in \Gamma(0,r)} |z g(z)| < 4r. \tag{11}$$

So for $w \in D = B(0,r)$ and $z \in \Omega$, (10) and (11) give that

$$|Q(z,w)| \leqslant |g(z)| + (|w| + |\alpha|)|g(z)|^2 < \frac{4}{r} + (r + 4r)(\frac{4}{r})^2 < \frac{100}{r}.$$

Hence we have proved (8).

Now fix $w \in D$ and define $\phi \colon \Omega \to \mathbb{C}$ by

$$\phi(z) = (Q(z,w) - \frac{1}{z-w})(z-w)^3$$

If $z \in \Omega \cap D$, then in particular $|z| < r$, so $|z-w| \leqslant |z| + |w| < 2r$, so (8) gives that

$$|\phi(z)| \leqslant \frac{100}{r}(2r)^3 + (2r)^2 < 1\,000r^2. \tag{12}$$

We shall use this result below.

Now fix $z \in \Omega$. Let $\beta = F^{-1}(z)$. Observe that

$$zg(z) = \frac{1}{a} zF^{-1}(z) = \frac{1}{a} F(\beta)\beta = 1 + a^{-1} \sum_{n=0}^{\infty} c_n \beta^{n+1}. \tag{13}$$

By our choice of F, then $\beta \to 0$ as $z \to \infty$, so (13) implies that $zg(z) \to 1$ as $z \to \infty$. Hence a Laurent expansion of g around any fixed $w \in D$ will be of the form:

$$g(z) = \frac{1}{z-w} + \frac{\lambda_2(w)}{(z-w)^2} + \frac{\lambda_3(w)}{(z-w)^3} + \cdots. \tag{14}$$

This expansion will be valid for large enough z. Let Γ be a circle centred on the origin, radius sufficiently large. Then by Cauchy's Residue Theorem,

$$\begin{aligned}
\lambda_2(w) &= \frac{1}{2\pi \mathrm{i}} \oint_\Gamma (z-w) g(z) \, \mathrm{d}z \\
&= \frac{1}{2\pi \mathrm{i}} \oint_\Gamma zg(z) \, \mathrm{d}z - \frac{w}{2\pi \mathrm{i}} \oint_\Gamma g(z) \, \mathrm{d}z \\
&= \alpha - w.
\end{aligned} \tag{15}$$

Now ϕ equals:

$$\begin{aligned}
\phi(z) &= (Q(z,w) - \frac{1}{z-w})(z-w)^3 \\
&= (g(z) + (w-\alpha)(g(z))^2 - \frac{1}{z-w})(z-w)^3.
\end{aligned}$$

So consider when z is large. By expanding g in the above expression according to (14), and then applying (15), we see that ϕ is bounded as $z \to \infty$. Hence ϕ has a removable singularity at ∞. Hence the maximum modulus principle and (12) give that $|\phi(z)| < 1\,000r^2$ for all $z \in \Omega$. Hence

$$\left| Q(z,w) - \frac{1}{z-w} \right| = \frac{|\phi(z)|}{|z-w|^3} < \frac{1\,000r^2}{|z-w|^3},$$

which is (9). □

Theorem 3.3.4 (Mergelyan's Theorem)**.** *Let $K \subseteq \mathbb{C}$ be compact, and such that $\mathbb{C} \setminus K$ is connected. Then $f \in P(K)$ if and only if $f \in A(K)$.*

Proof. The 'only if' direction is trivial: if f is a uniform limit of polynomials on K, then it is a uniform limit of functions which are continuous in K and holomorphic in K°, and hence it must be also.

Conversely suppose that $f \in A(K)$. By the Tietze Extension Theorem (Theorem 2.3.7), we may continuously extend f to $\mathbb{C}$ with compact support: denote this extension again by f. Let ω be the modulus of continuity for f. As f has compact support, it is uniformly continuous, and hence $\omega(\delta) \to 0$ as $\delta \to 0$. We will show for any $\delta > 0$ that there is a polynomial P such that for all $z \in K$,

$$|f(z) - P(z)| < 10\,000\,\omega(\delta).$$

Throughout the remainder of this proof, fix $\delta > 0$.

With S as in Lemma 3.3.1, and $w = \zeta + \mathrm{i}\eta$, define Φ by

$$\Phi(z) = \iint_{\mathbb{C}} f(z-w)S(w)\,\mathrm{d}\zeta\,\mathrm{d}\eta.$$

As both f and S have compact support, Φ has compact support also.[2] Using properties (4), (5) in Lemma 3.3.1, we have for all $z \in \mathbb{C}$ that

$$\begin{aligned} |\Phi(z) - f(z)| &= \left| \iint_{\mathbb{C}} f(z-w)S(w)\,\mathrm{d}\zeta\,\mathrm{d}\eta - \iint_{\mathbb{C}} f(z)S(w)\,\mathrm{d}\zeta\,\mathrm{d}\eta \right| \\ &\leqslant \iint_{\mathbb{C}} |f(z-w) - f(z)|S(w)\,\mathrm{d}\zeta\,\mathrm{d}\eta \\ &\leqslant \omega(\delta) \iint_{\mathbb{C}} S(w)\,\mathrm{d}\zeta\,\mathrm{d}\eta \\ &= \omega(\delta). \end{aligned} \tag{16}$$

Next, the difference quotients of S converge boundedly to its partial derivatives, as S has compact support. So by the Dominated Convergence Theorem, letting ∂ represent any of $\partial_{\bar{z}}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$,

$$\begin{aligned} \partial\Phi(z) &= \partial \iint_{\mathbb{C}} f(z-w)S(w)\,\mathrm{d}\zeta\,\mathrm{d}\eta \\ &= \partial \iint_{\mathbb{C}} f(w)S(z-w)\,\mathrm{d}\zeta\,\mathrm{d}\eta \\ &= \iint_{\mathbb{C}} f(w)(\partial S)(z-w)\,\mathrm{d}\zeta\,\mathrm{d}\eta \\ &= \iint_{\mathbb{C}} f(z-w)\partial S(w)\,\mathrm{d}\zeta\,\mathrm{d}\eta. \end{aligned}$$

Hence Φ has continuous partial derivatives, and so $\Phi \in C_c^1(\mathbb{C})$. Furthermore,

$$\begin{aligned} |\partial_{\bar{z}}\Phi(z)| &= \left| \iint_{\mathbb{C}} f(z-w)\partial_{\bar{z}}S(w)\,\mathrm{d}\zeta\,\mathrm{d}\eta \right| \\ &= \left| \iint_{\mathbb{C}} (f(z-w) - f(z))\partial_{\bar{z}}S(w)\,\mathrm{d}\zeta\,\mathrm{d}\eta \right| \\ &\leqslant \iint_{\mathbb{C}} |f(z-w) - f(z)||\partial_{\bar{z}}S(w)|\,\mathrm{d}\zeta\,\mathrm{d}\eta \\ &< \frac{2\omega(\delta)}{\delta}, \end{aligned} \tag{17}$$

[2]The function Φ may be regarded as a 'smoothing out' (a mollification) of f. We shall see that Φ equals f sufficiently far within K (where f is holomorphic), agreeing with our intuition that a holomorphic function should already be, in some non-technical sense, 'smooth'.

where the second line follows from (6) in Lemma 3.3.1, and the last line follows from (4) and (7) in Lemma 3.3.1. This statement might be interpreted as meaning that Φ is 'almost holomorphic'.

We will now make good on our earlier footnote by showing that $\Phi(z) = f(z)$ for those $z \in K$ such that $\operatorname{dist}(z, \mathbb{C} \setminus K) > \delta$. Letting $w = r\mathrm{e}^{\mathrm{i}\theta}$, using (4) in Lemma 3.3.1, and using Cauchy's Formula (valid as $\operatorname{dist}(z, \mathbb{C} \setminus K) > \delta$ implies that f is holomorphic in a neighbourhood of $\overline{B}(z, \delta)$),

$$\begin{aligned}
\Phi(z) &= \iint_{\mathbb{C}} f(z-w) S(w) \,\mathrm{d}\zeta \,\mathrm{d}\eta \\
&= \int_0^{2\pi} \int_0^{\delta} f(z - r\mathrm{e}^{\mathrm{i}\theta}) S(r\mathrm{e}^{\mathrm{i}\theta}) r \,\mathrm{d}r \,\mathrm{d}\theta \\
&= \int_0^{\delta} s(r) r \int_0^{2\pi} f(z - r\mathrm{e}^{\mathrm{i}\theta}) \,\mathrm{d}\theta \,\mathrm{d}r \\
&= \int_0^{\delta} s(r) r \; 2\pi f(z) \;\mathrm{d}r \\
&= f(z) \iint_{\mathbb{C}} S(w) \,\mathrm{d}\zeta \,\mathrm{d}\eta \\
&= f(z).
\end{aligned}$$

In particular this shows that Φ is holomorphic sufficiently far within K, and hence that $\partial_{\bar{z}} \Phi = 0$ there. Along with Lemma 2.3.1 (as we have already shown that $\Phi \in C_c^1(\mathbb{C})$), this implies that

$$\begin{aligned}
\Phi(z) &= -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{(\partial_{\bar{z}} \Phi)(w)}{w - z} \,\mathrm{d}\zeta \,\mathrm{d}\eta \\
&= -\frac{1}{\pi} \iint_{X} \frac{(\partial_{\bar{z}} \Phi)(w)}{w - z} \,\mathrm{d}\zeta \,\mathrm{d}\eta, \qquad (18)
\end{aligned}$$

where $X = \{z \in \operatorname{supp}(\Phi) : \operatorname{dist}(z, \mathbb{C} \setminus K) \leqslant \delta\}$. The definition of X shows that X is compact, and may be covered by finitely many open discs $D_1, \ldots, D_n$ of radius 2δ, whose centres are not in K. As their centres are not in the closed K, there exist closed discs d_j, concentric with D_j, which are disjoint from K.

Furthermore, as $\mathbb{C} \setminus K$ is connected, there exists a polygonal path[3] in $\mathbb{C} \setminus K$ from the centre of each D_j to ∞. By taking a restriction of this, we may find a simple path γ_j in $\mathbb{C} \setminus K$ from some point on ∂d_j to some point on ∂D_j. Let $E_j = d_j \cup \gamma_j$. This set is compact, simply connected, disjoint from K, and of diameter at least 2δ.

So by applying Lemma 3.3.3 with $r = 2\delta$, $D = D_j$, and $E = E_j$, there must exist functions $g_j \in H(\widehat{\mathbb{C}} \setminus E_j)$ and constants $\alpha_j \in \mathbb{C}$, such that for all $z \in \widehat{\mathbb{C}} \setminus E_j$ and all $w \in D_j$,

[3] That $U \subseteq \mathbb{C}$ is open and connected implies that U is polygonally path connected: fix any $x \in U$. Openness implies that the set of points which can be polygonally path connected to x is open. Similarly for its complement. These partition the set. As the first set contains x, it is nonempty, and hence connectedness implies that it is the whole set.

$$|Q_j(z,w)| < \frac{50}{\delta}, \tag{19}$$

$$\left|Q_j(z,w) - \frac{1}{z-w}\right| < \frac{4\,000\delta^2}{|z-w|^3}, \tag{20}$$

where $Q_j(z,w) = g_j(z) + (w-\alpha)(g_j(z))^2$.

Now let $X_1 = X \cap D_1$, and for $2 \leqslant j \leqslant n$ let $X_j = (X \cap D_j) \setminus (X_1 \cup \cdots \cup X_{j-1})$, so that $\{X_1, \ldots, X_n\}$ is a partition of X. Let $\Omega = \mathbb{C} \setminus (E_1 \cup \cdots \cup E_n)$, so Ω is an open set containing K. Now define $F\colon \Omega \to \mathbb{C}$ by

$$F(z) = \sum_{j=1}^{n} \frac{1}{\pi} \iint_{X_j} (\partial_{\bar{z}}\Phi)(w) Q_j(z,w) \,\mathrm{d}\zeta\,\mathrm{d}\eta.$$

The definition of Q_j means that F is a linear combination of g_j and g_j^2, and so $F \in H(\Omega)$. Furthermore, we have from (18) and (17) that

$$\begin{aligned}
|F(z) - \Phi(z)| &= \left|\sum_{j=1}^{n} \frac{1}{\pi} \iint_{X_j} (\partial_{\bar{z}}\Phi)(w) Q_j(z,w) \,\mathrm{d}\zeta\,\mathrm{d}\eta + \frac{1}{\pi} \iint_{X} \frac{(\partial_{\bar{z}}\Phi)(w)}{w-z} \,\mathrm{d}\zeta\,\mathrm{d}\eta\right| \\
&= \left|\sum_{j=1}^{n} \frac{1}{\pi} \iint_{X_j} (\partial_{\bar{z}}\Phi)(w) (Q_j(z,w) - \frac{1}{z-w}) \,\mathrm{d}\zeta\,\mathrm{d}\eta\right| \\
&\leqslant \sum_{j=1}^{n} \frac{1}{\pi} \iint_{X_j} |(\partial_{\bar{z}}\Phi)(w)| \left|Q_j(z,w) - \frac{1}{z-w}\right| \mathrm{d}\zeta\,\mathrm{d}\eta \\
&< \frac{2\omega(\delta)}{\pi\delta} \sum_{j=1}^{n} \iint_{X_j} \left|Q_j(z,w) - \frac{1}{z-w}\right| \mathrm{d}\zeta\,\mathrm{d}\eta.
\end{aligned}$$

Estimate each integral in this sum by letting $w = z + r\mathrm{e}^{\mathrm{i}\theta}$, and bounding the integrand by (19) for $r < 4\delta$ and by (20) for $r \geqslant 4\delta$, to get that

$$\begin{aligned}
&\sum_{j=1}^{n} \iint_{X_j} \left|Q_j(z,w) - \frac{1}{z-w}\right| \mathrm{d}\zeta\,\mathrm{d}\eta \\
&\qquad < \sum_{j=1}^{n} \left[\iint_{X_j \cap B(z,4\delta)} (\frac{50}{\delta} + \frac{1}{r}) \,\mathrm{d}\zeta\,\mathrm{d}\eta + \iint_{X_j \setminus B(z,4\delta)} \frac{4\,000\delta^2}{r^3} \,\mathrm{d}\zeta\,\mathrm{d}\eta \right] \\
&\qquad = \iint_{X \cap B(z,4\delta)} (\frac{50}{\delta} + \frac{1}{r}) \,\mathrm{d}\zeta\,\mathrm{d}\eta + \iint_{X \setminus B(z,4\delta)} \frac{4\,000\delta^2}{r^3} \,\mathrm{d}\zeta\,\mathrm{d}\eta \\
&\qquad \leqslant 2\pi \int_0^{4\delta} (\frac{50}{\delta} + \frac{1}{r}) r \,\mathrm{d}r + 2\pi \int_{4\delta}^{\infty} \frac{4\,000\delta^2}{r^3} r \,\mathrm{d}r \\
&\qquad < 3\,000\pi\delta.
\end{aligned}$$

Hence

$$|F(z) - \Phi(z)| < 6\,000\omega(\delta). \tag{21}$$

Now Ω is open and contains K. Furthermore, $F \in H(\Omega)$. So Runge's Theorem (Theorem 3.2.2) implies that there exists some polynomial P such that

$$|F(z) - P(z)| < \omega(\delta) \tag{22}$$

for all $z \in K$.

Finally, combining (16), (21) and (22) gives our result:

$$\begin{aligned} |f(z) - P(z)| &\leqslant |f(z) - \Phi(z)| + |\Phi(z) - F(z)| + |F(z) - P(z)| \\ &< \omega(\delta) + \omega(\delta) + 6\,000\omega(\delta) \\ &< 10\,000\omega(\delta). \end{aligned}$$

□

Example 3.3.5. Having now seen both Mergelyan's Theorem and Runge's Theorem, one might wonder if there is a shorter way between the two: in particular, is it true that for any function in $A(K)$ we may find Ω open and containing K such that $f \in H(\Omega)$? The answer is no.

Consider when $K = \overline{B}(0,1)$, and $f(z) = (z-1)\exp(1/(z-1))$. Clearly this is holomorphic in $B(0,1)$, and continuous in $\overline{B}(0,1) \setminus \{1\}$. Treating $1/(z-1)$ as a Möbius transformation, it follows that $\exp(1/(z-1))$ is bounded on $B(0,1)$; thus f is continuous at 1 (when approached from within $B(0,1)$). But f is not holomorphic on any neighbourhood of 1. In fact it is not even continuous on any neighbourhood of 1, as $f(z) \to 0$ as $z \to 1$ along the real axis to the left of 1, but $f(z) \to \infty$ as $z \to 1$ along the real axis to the right of 1.

4 Functional Analytic Proof

The complex analytic proof of Mergelyan's Theorem, whilst explicit and constructive, offers little insight into the structure of the broader theory. (Although not entirely; see the introduction to Section 3.3.) It is via functional analysis that Mergelyan's Theorem finds its natural home, as a problem on the function algebra $C(K)$. Our goal is to characterise its closed subspace $P(K)$, by showing that it equals $A(K)$.

The functional analytic proof that we present here has undergone several rounds of simplification. Bishop [6, Theorem 4] presents a proof in terms of 'analytic differentials', building on his previous paper [4].[4] As the theory progressed, Glicksberg and Wermer [14] substantially simplified the argument through abstract results on Dirichlet algebras. But this can actually be simplified further, away from the abstraction of Dirichlet algebras, allowing Carleson [8] to present a proof without introducing the notion; we shall not either. Even so, the structure of the argument of [14] remains apparent; it is worth comparing its proof to the one that we give here.[5]

[4] See [5] and [3] for some of Bishop's related work, but much of it has since been superseded.

[5] In fact, the greater generality of Dirichlet algebras (and other generalisations beyond them) serves to illuminate much of the further theory, some of which we shall see in Section 5. It is substantially more complicated however, and remains beyond the scope of this paper.

4.1 Functional Analytic Preliminaries

We will use several different measures in our proof, often simultaneously. It is tiresome to repeat 'almost everywhere with respect to μ', varying only which measure we are talking about. As such we shall abbreviate the statement to simply $[\mu]$ or 'μ-almost everywhere'. Similarly we shall abbreviate 'almost all, with respect to μ' to 'μ-almost all'. When we are referring to the Lebesgue measure, we shall use L in place of μ.

Now we state some preliminary definitions and results, without proof.

Definition 4.1.1. Let $K \subseteq \mathbb{C}$ be compact. A bounded linear functional $\phi: C(K) \to \mathbb{C}$ is *positive* if $\phi(f) \geqslant 0$ for all $f \in C(K)$ whose range lies in $[0, \infty)$.

Definition 4.1.2. Let $F(z) = \int f(z, w) \, \mathrm{d}\mu(w)$ be some integral. Then we say that F *converges absolutely* at z if $\int f(z, w) \, |\mathrm{d}\mu(w)| < \infty$. Else F is said to *diverge* at z.

The 'Riesz Representation Theorem' is a name which may refer to several different results, all in a similar vein. Here we present two particular variations, which are sometimes also known as the Riesz–Kakutani Representation Theorem or the Riesz–Markov Theorem.[6] They characterise the dual of the space of continuous functions on some compact set[7] as being the space of measures. The other common 'Riesz Representation Theorem', regarding the duals of Hilbert spaces, will also be used later. For proofs, see [21, Theorems 2.14 and 6.19], see also [19, Appendix A].

Theorem 4.1.3 (Riesz Representation Theorem, Real Case)**.** *Let $K \subseteq \mathbb{C}$ be compact, and let ϕ: $C_{\mathbb{R}}(K) \to \mathbb{R}$ be a bounded linear functional. Then there exists a unique finite real measure μ on K such that for all $f \in C(K)$,*

$$\phi(f) = \int_K f \, \mathrm{d}\mu.$$

Furthermore, $\|\phi\| = |\mu|(K)$.

Theorem 4.1.4 (Riesz Representation Theorem)**.** *Let $K \subseteq \mathbb{C}$ be compact, and let ϕ: $C(K) \to \mathbb{C}$ be a bounded linear functional. Then there exists a unique complex measure μ on K such that for all $f \in C(K)$,*

$$\phi(f) = \int_K f \, \mathrm{d}\mu.$$

Furthermore, $\|\phi\| = |\mu|(K)$. Additionally, if ϕ is positive, then μ is positive.

(That the measure should be finite is omitted from this case, as we recall that all complex measures are automatically of bounded variation.)

Remark 4.1.5. It follows that the space of complex measures is a Banach space when equipped with the norm $\|\mu\| = |\mu|(K)$.

[6] It is through exploiting these that we may simplify away from Dirichlet algebras; contrast with [14, Proposition 5].

[7] More generally, the space of functions on a locally compact Hausdorff space which vanish at infinity.

We will refer to a measure as being *associated* with a particular bounded linear functional, and vice versa. We shall refer to a measure as *annihilating* a subset of the continuous functions if its associated bounded linear functional does so.

Definition 4.1.6. Let $K \subseteq \mathbb{C}$ be compact, and let $a \in K$. Then the map $f \mapsto f(a)$ is a positive bounded linear functional on $C(K)$. So by the Riesz Representation Theorem, it has an associated finite positive measure δ_a on K, called the *Dirac measure.*[8]

Lastly we have a standard property of harmonic functions, see [16, Theorem 7.2.5], see also [21, p. 237].

Theorem 4.1.7 (Mean value property for harmonic functions)**.** *Let $z \in \mathbb{C}$ and $r > 0$. Let $u \in C(B(z,r))$ be harmonic. Then, with $w = \zeta + \mathrm{i}\eta$,*

$$u(z) = \frac{1}{\pi r^2} \iint_{B(z,r)} u(w) \, \mathrm{d}\zeta \, \mathrm{d}\eta.$$

4.2 Lebesgue Decomposition Theorem

Being able to decompose those measures we are interested in will prove useful in analysing their behaviour. We initially follow [21, Proposition 6.8]. Our proof of the Lebesgue Decomposition Theorem itself is adapted from both [17, Section 32, Theorem C] and [21, Theorem 6.10] in order to give the shortest possible proof of the result, without needing to mention the Radon–Nikodym Theorem.

Definition 4.2.1. Let $(X, \mathcal{M})$ be a measurable space. Let μ be a positive or complex measure and α be a positive measure on $(X, \mathcal{M})$. Then μ is *absolutely continuous* with respect to α if every α-null set is also μ-null. We denote this by $\mu \ll \alpha$.

Definition 4.2.2. Let $(X, \mathcal{M})$ be a measurable space. Let μ be a positive or complex measure on $(X, \mathcal{M})$. Then μ is *concentrated* on $X \in \mathcal{M}$ if for all $E \in \mathcal{M}$ we have that $\mu(E) = \mu(E \cap X)$.

Definition 4.2.3. Let $(X, \mathcal{M})$ be a measurable space. Let μ, ν be two positive or complex measures on $(X, \mathcal{M})$. Then μ and ν are *mutually singular* if there exist disjoint $Y, Z \in \mathcal{M}$ such that μ is concentrated on Y and ν is concentrated on Z. We denote this by $\mu \perp \nu$.

Definition 4.2.4. Let $(X, \mathcal{M})$ be a measurable space. Let μ be a positive or complex measure and α be a positive measure on $(X, \mathcal{M})$. Then $\mu = \mu_a + \mu_s$ is a *Lebesgue decomposition* of μ with respect to α, where μ_a, μ_s are complex measures on $(X, \mathcal{M})$ such that $\mu_a \ll \alpha$, and $\mu_s \perp \alpha$.

Lemma 4.2.5. *Let $(X, \mathcal{M})$ be a measurable space. Let μ, μ_1, μ_2 be complex measures and α, β be positive measures on $(X, \mathcal{M})$.*

(i) *If $\mu_1 \ll \alpha$ and $\mu_2 \ll \alpha$ then $\mu_1 + \mu_2 \ll \alpha$.*

(ii) *If $\mu_1 \perp \alpha$ and $\mu_2 \perp \alpha$ then $\mu_1 + \mu_2 \perp \alpha$.*

[8]It is given by $\delta_a(E) = \mathbb{1}_E(a)$.

(iii) *If* $\mu \ll \alpha$ *and* $\mu \perp \alpha$ *then* $\mu = 0$.

(iv) *If* $\mu \ll \alpha$ *and* $\alpha \ll \beta$ *then* $\mu \ll \beta$.

(v) *If* $\mu \perp \alpha$ *and* $\beta \ll \alpha$ *then* $\mu \perp \beta$.

Proof.

(i) This is immediate from the definition of absolute continuity.

(ii) Let $Y_1, Z_1 \in \mathcal{M}$ be disjoint sets such that μ_1 is concentrated on Y_1 and α is concentrated on Z_1. Similarly let $Y_2, Z_2 \in \mathcal{M}$ be disjoint sets such that μ_2 is concentrated on Y_2 and α is concentrated on Z_2. Then $\mu_1 + \mu_2$ is concentrated on $Y_1 \cup Y_2$ and α is concentrated on $Z_1 \cap Z_2$, which are disjoint.

(iii) By mutual singularity, let $Y, Z \in \mathcal{M}$ be disjoint sets such that μ is concentrated on Y and α is concentrated on Z. Then $X \setminus Z$ is α-null, so $Y \subseteq X \setminus Z$ is α-null, and so all measurable $F \subseteq Y$ are α-null. By absolute continuity, all measurable $F \subseteq Y$ are μ-null. But then for any $E \in \mathcal{M}$, we have that $\mu(E) = \mu(E \cap Y) = 0$.

(iv) This is immediate from the definition of absolute continuity.

(v) Let $Y, Z \in \mathcal{M}$ be disjoint sets such that μ is concentrated on Y and α is concentrated on Z. Then $X \setminus Z$ is α-null, and hence $X \setminus Z$ is β-null. Hence β is concentrated on Z.

□

Theorem 4.2.6 (Lebesgue Decomposition Theorem)**.** *Let* $(X, \mathcal{M})$ *be a measurable space. Let* μ *be a complex measure and* α *be a finite positive measure on* $(X, \mathcal{M})$*. Then there exists a unique Lebesgue decomposition of* μ *with respect to* α*. Furthermore, the decomposition is* concentration preserving*, meaning that if* μ *is concentrated on* $S \in \mathcal{M}$*, then both parts of the decomposition are also concentrated on* S*.*

Proof. First assume that μ is a finite positive measure. Then $\nu = \mu + \alpha$ is a finite positive measure, and so $L^2(X, \mathcal{M}, \nu)$ is a Hilbert space. Hence for $f \in L^2(X, \mathcal{M}, \nu)$, the Cauchy–Schwarz inequality gives that

$$\left| \int_X f \,\mathrm{d}\mu \right| \leqslant \int_X |f| \,\mathrm{d}\mu \leqslant \int_X |f| \,\mathrm{d}\nu \leqslant \left(\int_X |f|^2 \,\mathrm{d}\nu \right)^{\frac{1}{2}} (\nu(X))^{\frac{1}{2}}.$$

So as ν is finite, $f \mapsto \int_X f \,\mathrm{d}\mu$ is a bounded linear functional on $L^2(X, \mathcal{M}, \nu)$. The Riesz Representation Theorem (the Hilbert spaces version) then implies that there exists $g \in L^2(X, \mathcal{M}, \nu)$ such that

$$\int_X f \,\mathrm{d}\mu = \int_X fg \,\mathrm{d}\nu.$$

So by taking f to be the indicator function of any $E \in \mathcal{M}$,

$$\mu(E) = \int_E g \,\mathrm{d}\nu. \tag{23}$$

As $0 \leqslant \mu(E) \leqslant \nu(E)$, substitute in (23) to get that $0 \leqslant g \leqslant 1$ $[\nu]$. As $\mu \ll \nu$, then $0 \leqslant g \leqslant 1$ $[\mu]$.

Now let $Y = \{x \in X : g(x) = 1\}$ and $Z = \{x \in X : 0 \leqslant g(x) < 1\}$. (These are defined up to a μ-null set; pick any representative of g to define them.) Then define μ_a and μ_s by

$$\begin{aligned} \mu_a(E) &= \mu(E \cap Z), \\ \mu_s(E) &= \mu(E \cap Y). \end{aligned}$$

It is immediate from these definitions that the decomposition is concentration preserving.

Now μ_s is concentrated on Y. Furthermore, by (23) and the definition of Y,

$$\mu(Y) = \int_Y \mathrm{d}\nu = \mu(Y) + \alpha(Y)$$

and so $\alpha(Y) = 0$, since μ is finite. Hence α is concentrated on $X \setminus Y$, and so $\mu_s \perp \alpha$.

It remains to show that $\mu_a \ll \alpha$. Let $E \in \mathcal{M}$ be α-null. We will show that E is μ_a-null. By (23),

$$\int_{E\cap Z} \mathrm{d}\mu = \mu(E \cap Z) = \int_{E\cap Z} g\,\mathrm{d}\nu = \int_{E\cap Z} g\,\mathrm{d}\mu + \int_{E\cap Z} g\,\mathrm{d}\alpha = \int_{E\cap Z} g\,\mathrm{d}\mu.$$

Hence

$$\int_{E\cap Z} (1-g)\,\mathrm{d}\mu = 0.$$

Since $1 - g \geqslant 0$ $[\mu]$, this implies that $\mu_a(E) = \mu(E \cap Z) = 0$, that is, E is μ_a-null.

Now consider when μ is not necessarily a finite positive measure. It decomposes into $\mu = \mu_r + \mathrm{i}\mu_i$, with μ_r and μ_i real measures. In turn decompose μ_r and μ_i into their positive and negative variations (that is, their Jordan decompositions). As μ is a complex measure, and hence is of bounded variation, these positive and negative variations will be finite positive measures. Hence we may apply the result as it has just been proved to each of them, and then use parts (i) and (ii) of Lemma 4.2.5 to assemble their Lebesgue decompositions into a Lebesgue decomposition for μ.

Finally we show that Lebesgue decompositions are unique: suppose that $\mu = \mu_a + \mu_s$ and $\mu = \widehat{\mu}_a + \widehat{\mu}_s$ are two Lebesgue decompositions of μ with respect to α. Then $\mu_a - \widehat{\mu}_a = \mu_s - \widehat{\mu}_s$. Part (i) of Lemma 4.2.5 gives that the left hand side is absolutely continuous with respect to α, whilst part (ii) gives that the right hand side is mutually singular with respect to α. Hence part (iii) gives that both sides are zero, and thus that the decomposition is unique. □

We finish with a result that is unrelated to the Lebesgue Decomposition Theorem, but whose content is essentially measure-theoretic.

Lemma 4.2.7. *Let $K \subseteq \mathbb{C}$ be compact, and let μ_1 and μ_2 be mutually singular complex measures on K. Then $\|\mu_1 + \mu_2\| = \|\mu_1\| + \|\mu_2\|$. (Recalling that $\|\mu\| = |\mu|(K)$.)*

Proof. Recalling Remark 4.1.5, that $\|\cdot\|$ is a norm on the space of complex measures, then certainly $\|\mu_1 + \mu_2\| \leqslant \|\mu_1\| + \|\mu_2\|$. Conversely, let $Y, Z \subseteq K$ be disjoint and measurable such that μ_1 is concentrated on Y and μ_2 is concentrated on $Z \subseteq K \setminus Y$. Let $\{K_i\}$ and $\{L_i\}$ be any partitions of K. Then $\{K_i \cap Y\} \cup \{L_i \setminus Y\}$ is also a partition of K, and hence

$$\begin{aligned}\sum_{i=1}^{\infty} |\mu_1(K_i)| + \sum_{i=1}^{\infty} |\mu_2(L_i)| &= \sum_{i=1}^{\infty} |\mu_1(K_i \cap Y)| + \sum_{i=1}^{\infty} |\mu_2(L_i \setminus Y)| \\ &= \sum_{i=1}^{\infty} |(\mu_1 + \mu_2)(K_i \cap Y)| + \sum_{i=1}^{\infty} |(\mu_1 + \mu_2)(L_i \setminus Y)| \\ &\leqslant \sup \sum_{i=1}^{\infty} |(\mu_1 + \mu_2)(M_i)| \\ &= \|\mu_1 + \mu_2\|\end{aligned}$$

where the sup is over all partitions $\{M_i\}$ of K.

Now sup over all partitions $\{K_i\}$, $\{L_i\}$ of K to deduce the result. □

4.3 Walsh–Lebesgue Theorem

The Walsh–Lebesgue Theorem is a now-classical theorem due to Walsh, see [24] and [25], and may be regarded as a 'real version' of Mergelyan's Theorem, as it can be interpreted as dealing with the uniform approximation of harmonic functions, continuous up to the boundary, by the real parts of polynomials.

We shall prove the result via logarithmic potentials. These have a deep theory all of their own, in particular to potential problems in the plane. They also turn out to be directly relevant to the problem of polynomial approximation, see [22]. We only skirt over the theory here – see [23] for a further discussion of logarithmic potentials, and [2, Chapter X, Section 7] for an advanced take (involving logarithmic capacity) on how they interact with the present problem. It is from [23, Chapter 0, Theorem 5.6] that we get Lemma 4.3.7, otherwise we largely work from [8, Lemmas 1–3], which presents a concise proof of the result.

Definition 4.3.1. Let $K \subseteq \mathbb{C}$ be compact. Let α be a finite real measure on K. Define $u\colon \mathbb{C} \to \overline{\mathbb{R}}$ by

$$u(z) = \int_K \log \left| \frac{1}{z - w} \right| \mathrm{d}\alpha(w).$$

Then $u(z)$ is said to be the *logarithmic potential* with respect to the measure α.

Lemma 4.3.2. *Let $K \subseteq \mathbb{C}$ be compact. Let u be the logarithmic potential with respect to α, a finite real measure on K. Then u converges absolutely L-almost everywhere in $\mathbb{C}$. Let Ω be a connected component of $\mathbb{C} \setminus K$. If $u(z) = 0$ for all $z \in \Omega$, then $u(z) = 0$ for all $z \in \overline{\Omega}$ at which u converges absolutely.*

Proof. Let v be the logarithmic potential with respect to $|\alpha|$. Pick any $R > 0$. Let $S = R + \sup_{w \in K} |w|$. Let $z = x + \mathrm{i}y = w + r\mathrm{e}^{\mathrm{i}\theta}$. Then by applying Fubini's Theorem (justified as our eventual result is finite),

$$\begin{aligned}
\iint_{B(0,R)} v(z)\,\mathrm{d}x\,\mathrm{d}y &= \iint_{B(0,R)} \int_K \log\left|\frac{1}{z-w}\right| |\mathrm{d}\alpha(w)|\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant \int_K \iint_{B(w,S)} \log\left|\frac{1}{z-w}\right| \mathrm{d}x\,\mathrm{d}y\,|\mathrm{d}\alpha(w)| \\
&= \int_K |\mathrm{d}\alpha(w)| \int_0^{2\pi}\int_0^S \log\left(\frac{1}{r}\right) r\,\mathrm{d}r\,\mathrm{d}\theta \\
&< \infty.
\end{aligned}$$

Hence $v(z) < \infty$ for L-almost all $z \in B(0,R)$. As R was arbitrary, u converges absolutely L-almost everywhere in $\mathbb{C}$.

For the next part, suppose that $u(z) = 0$ for all $z \in \Omega$, and that u converges absolutely at $z_0 \in \partial\Omega$. We will show that $u(z_0) = 0$. Without loss of generality, we let $z_0 = 0$. This is just to simplify the notation, as the next part is already very technical.

For $\delta \in (0, \frac{1}{2})$, let $D(\delta) = \overline{\Omega \cap B(0,\delta)}$. Define $k_\delta \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ by

$$k_\delta(r) = \int_0^{2\pi} \mathbb{1}_{D(\delta)}(re^{\mathrm{i}\theta})\,\mathrm{d}\theta$$

and thus define $\Lambda_\delta \colon C(D(\delta)) \to \mathbb{C}$ by

$$\Lambda_\delta(f) = \iint_{D(\delta)} \frac{f(z)}{k_\delta(r)}\,\mathrm{d}\theta\,\mathrm{d}r,$$

where $z = x + \mathrm{i}y = re^{\mathrm{i}\theta}$ as usual. Now for integrable functions f depending only on r,

$$\begin{aligned}
\Lambda_\delta(f(r)) &= \iint_{D(\delta)} \frac{f(r)}{k_\delta(r)}\,\mathrm{d}\theta\,\mathrm{d}r \\
&= \int_0^\delta \frac{f(r)}{k_\delta(r)} \int_0^{2\pi} \mathbb{1}_{D(\delta)}\,\mathrm{d}\theta\,\mathrm{d}r \\
&= \int_0^\delta f(r)\,\mathrm{d}r. \qquad (24)
\end{aligned}$$

Hence in particular (now for any integrable f), as the constant function 1 is radial,

$$|\Lambda(f)| \leqslant \Lambda(1) \sup_{z\in D(\delta)} |f(z)| = \delta \sup_{z\in D(\delta)} |f(z)|$$

So Λ is a positive bounded linear functional on $C(D(\delta))$. Hence by the Riesz Representation Theorem (Theorem 4.1.4), Λ_δ will have some associated finite positive measure σ_δ on $D(\delta)$. So by (24), for integrable functions f depending only on r,

$$\int_{D(\delta)} f(r)\,\mathrm{d}\sigma_\delta = \Lambda_\delta(f(r)) = \int_0^\delta f(r)\,\mathrm{d}r.$$

By this property of σ_δ,

$$\begin{aligned}
\frac{1}{\delta}\int_{D(\delta)} \log\left|\frac{1}{z-w}\right| \mathrm{d}\sigma_\delta(z) &\leqslant \frac{1}{\delta}\int_{D(\delta)} \log\left|\frac{1}{|z|-|w|}\right| \mathrm{d}\sigma_\delta(z) \\
&= \frac{1}{\delta}\int_0^\delta \log\left|\frac{1}{r-|w|}\right| \mathrm{d}r \\
&= \log\left|\frac{1}{w}\right| + \frac{1}{\delta}\int_0^\delta \log\left|\frac{1}{1-r/|w|}\right| \mathrm{d}r \\
&\leqslant \log\left|\frac{1}{w}\right| + C, \qquad (25)
\end{aligned}$$

where $C = \sup_{T>0}\left[\frac{1}{T}\int_0^T \log|1/(1-t)|\,\mathrm{d}t\right]$.

Now fix $\rho \in (0, \frac{1}{2})$. Then by continuity of the integrand,

$$\frac{1}{\delta}\int_{D(\delta)} \log\left|\frac{1}{z-w}\right| \mathrm{d}\sigma_\delta(z) \longrightarrow \log\left|\frac{1}{w}\right| \qquad (26)$$

uniformly over $|w| > \rho$ as $\delta \to 0$.

As $u(z) = 0$ for $z \in \Omega$, by Fubini's Theorem,

$$\begin{aligned}
0 &= \frac{1}{\delta}\int_{D(\delta)} u(z)\,\mathrm{d}\sigma_\delta(z) \\
&= \frac{1}{\delta}\int_{D(\delta)}\int_K \log\left|\frac{1}{z-w}\right| \mathrm{d}\alpha(w)\,\mathrm{d}\sigma_\delta(z) \\
&= \int_{K\cap B(0,\rho)} \frac{1}{\delta}\int_{D(\delta)} \log\left|\frac{1}{z-w}\right| \mathrm{d}\sigma_\delta(z)\,\mathrm{d}\alpha(w) \\
&\quad + \int_{K\setminus B(0,\rho)} \frac{1}{\delta}\int_{D(\delta)} \log\left|\frac{1}{z-w}\right| \mathrm{d}\sigma_\delta(z)\,\mathrm{d}\alpha(w).
\end{aligned}$$

So rearrange this and apply (25) to obtain

$$\begin{aligned}
&\left|\int_{K\setminus B(0,\rho)} \frac{1}{\delta}\int_{D(\delta)} \log\left|\frac{1}{z-w}\right| \mathrm{d}\sigma_\delta(z)\,\mathrm{d}\alpha(w)\right| \\
&\qquad = \left|\int_{K\cap B(0,\rho)} \frac{1}{\delta}\int_{D(\delta)} \log\left|\frac{1}{z-w}\right| \mathrm{d}\sigma_\delta(z)\,\mathrm{d}\alpha(w)\right| \\
&\qquad \leqslant \int_{K\cap B(0,\rho)} \frac{1}{\delta}\int_{D(\delta)} \log\left|\frac{1}{z-w}\right| \mathrm{d}\sigma_\delta(z)\,|\mathrm{d}\alpha(w)| \\
&\qquad \leqslant \int_{K\cap B(0,\rho)} (\log\left|\frac{1}{w}\right| + C)\,|\mathrm{d}\alpha(w)|,
\end{aligned}$$

with the second inequality following because $\delta < \frac{1}{2}$, $\rho < \frac{1}{2}$ implies that $|z| < \frac{1}{2}$, $|w| < \frac{1}{2}$, and hence that $\log|1/(z-w)| > 0$.

Now take $\delta \to 0$ and apply (26) to obtain that

$$\left|\int_{K\setminus B(0,\rho)} \log\left|\frac{1}{w}\right| \mathrm{d}\alpha(w)\right| \leqslant \int_{K\cap B(0,\rho)} (\log\left|\frac{1}{w}\right| + C)\,|\mathrm{d}\alpha(w)|.$$

So recalling that u converges absolutely at $z_0 = 0$, take $\rho \to 0$ to obtain the result. □

Remark 4.3.3. When we use this result later, we will use it on both a compact set K, and on its boundary ∂K, which is also a compact set.

Lemma 4.3.4. *Let $\Phi \in C_c^2(\mathbb{C})$. Then the following formula holds, with $w = \zeta + \mathrm{i}\eta$:*

$$\Phi(z) = -\frac{1}{2\pi} \iint_{\mathbb{C}} (\Delta\Phi)(w) \log\left|\frac{1}{z-w}\right| \mathrm{d}\zeta\, \mathrm{d}\eta.$$

Proof. Fix z and let $w = z + r\mathrm{e}^{\mathrm{i}\theta}$. Then using the polar description of Δ,

$$\begin{aligned}
-\frac{1}{2\pi} &\iint_{\mathbb{C}} (\Delta\Phi)(w) \log\left|\frac{1}{z-w}\right| \mathrm{d}\zeta\, \mathrm{d}\eta \\
&= -\frac{1}{2\pi} \iint_{\mathbb{C}} (\Delta\Phi)(w) \log\left(\frac{1}{r}\right) r\, \mathrm{d}r\, \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} \left(\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2\Phi}{\partial\theta^2}\right)(w) \log(r)\, \mathrm{d}r\, \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} \frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right)(w) \log(r)\, \mathrm{d}r\, \mathrm{d}\theta \\
&= \Phi(z).
\end{aligned}$$

The third equality uses that Φ is 2π periodic in θ to conclude that the integral of $\frac{\partial^2\Phi}{\partial\theta^2}$ is zero. The fourth equality hides a routine calculation involving integration by parts. □

Lemma 4.3.5. *Let $K \subseteq \mathbb{C}$ be compact. Let u be the logarithmic potential with respect to α, a finite real measure on K. If $u(z) = 0$ for L-almost all $z \in \mathbb{C}$, then $\alpha \equiv 0$.*

Proof. Pick any $\Phi \in C_{\mathbb{R}}^2(K)$. Then $\Phi \in C_c^2(\mathbb{C})$, so apply Lemma 4.3.4 and Fubini's Theorem to obtain, with $w = \zeta + \mathrm{i}\eta$, that:

$$\begin{aligned}
\int_K \Phi(z)\, \mathrm{d}\alpha(z) &= -\frac{1}{2\pi} \int_K \left[\iint_{\mathbb{C}} (\Delta\Phi)(w) \log\left|\frac{1}{z-w}\right| \mathrm{d}\zeta\, \mathrm{d}\eta\right] \mathrm{d}\alpha(z) \\
&= -\frac{1}{2\pi} \iint_{\mathbb{C}} (\Delta\Phi)(w) \left[\int_K \log\left|\frac{1}{z-w}\right| \mathrm{d}\alpha(z)\right] \mathrm{d}\zeta\, \mathrm{d}\eta \\
&= -\frac{1}{2\pi} \iint_{\mathbb{C}} (\Delta\Phi)(w) u(z)\, \mathrm{d}\zeta\, \mathrm{d}\eta \\
&= 0.
\end{aligned}$$

Now $C_{\mathbb{R}}^2(K)$ is dense in $C_{\mathbb{R}}(K)$, hence the bounded linear map $f \mapsto \int_K f(z)\, \mathrm{d}\alpha(z)$ on $C_{\mathbb{R}}(K)$ is the zero map. Thus the zero measure may be associated with this map and so, as the real case of the Riesz Representation Theorem (Theorem 4.1.3) asserts that associated measures are unique, $\alpha \equiv 0$. □

Remark 4.3.6. A particularly interesting immediate consequence of the preceding lemma is that a measure is determined by its logarithmic potential.

Lemma 4.3.7. *Let $K \subseteq \mathbb{C}$ be compact. Let u be the logarithmic potential with respect to α, a finite real measure on K. Then u is harmonic in $\mathbb{C} \setminus K$, and is such that for all $z_0 \in \mathbb{C}$,*

$$u(z_0) \leqslant \varliminf_{z \to z_0} u(z).$$

Proof. As the integrand of $u(z)$ is harmonic for $z \in \mathbb{C} \setminus K$,

$$\Delta u(z) = \int_K \Delta \log \left| \frac{1}{z-w} \right| \mathrm{d}\alpha(w) = 0$$

by the Dominated Convergence Theorem, and hence u is harmonic in $\mathbb{C} \setminus K$.

Now fix $z_0 \in \mathbb{C}$. By the Monotone Convergence Theorem, for all $z \in \mathbb{C}$,

$$u(z) = \lim_{M \to \infty} u_M(z)$$

where

$$u_M(z) = \int_K \min\{M, \log \left| \frac{1}{z-w} \right|\} \mathrm{d}\alpha(w).$$

The integrand of u_M is continuous, and so every u_M is as well. Furthermore, $(u_M(z))$ is an increasing sequence, for all $z \in \mathbb{C}$. Hence $u_M(z) \leqslant u(z)$. Let $z \to z_0$ to obtain that $u_M(z_0) \leqslant \lim_{z \to z_0} u(z)$, and then let $M \to \infty$ to obtain that $u(z_0) \leqslant \lim_{z \to z_0} u(z)$. □

Remark 4.3.8. By taking a $\liminf_{z \to z_0}$ instead of a $\lim_{z \to z_0}$, we can actually deduce a stronger result about u, namely that it is *lower semicontinuous* in $\mathbb{C}$. In fact, it also has additional properties which make it *superharmonic*, see [23, Chapter 0, Theorem 5.6].

Theorem 4.3.9 (Walsh–Lebesgue Theorem)**.** *Let $K \subseteq \mathbb{C}$ be compact, and such that $\mathbb{C} \setminus K$ is connected. Let $\phi \in C_{\mathbb{R}}(\partial K)$. Then there exists a sequence of polynomials in z, call them (P_n), such that $(\mathrm{Re}(P_n))$ converges uniformly on ∂K to ϕ.*

Proof. Let $\mathrm{Re}\, P(\partial K)$ denote the set of all functions $f\colon \partial K \to \mathbb{R}$ which are uniform limits of the real parts of polynomials in z. Let α be a finite real measure on ∂K annihilating $\mathrm{Re}\, P(\partial K)$. As the imaginary part of any polynomial is the real part of another polynomial, α in fact annihilates $P(\partial K)$.

Let u be the logarithmic potential with respect to α. Let $D \subseteq \mathbb{C} \setminus K$ be any open disc. The integrand of u is harmonic in D, so it is the real part of a function holomorphic in D; this function will have a Taylor expansion in D. As α is real and the convergence of the Taylor expansion is uniform, both the Re and the summation may be commuted with the integral over ∂K (which is with respect to α), to conclude that u is zero in D. Hence by the identity theorem for harmonic functions, u is zero in $\mathbb{C} \setminus K$.

So by applying Lemma 4.3.2 to the compact set ∂K (with Ω equal to $\mathbb{C} \setminus K$), we know that u is zero L-almost everywhere on ∂K.

We now wish to show that u is zero for L-almost all points in K°. If K° is empty then this is vacuously true. Else, let Φ be the subspace of $C_{\mathbb{R}}(\partial K)$ consisting of those functions which have harmonic extensions to K°. By the maximum principle, any harmonic extension is unique, and so we we identify $\phi \in \Phi$ with its extension. Let $a \in K^\circ$. Then $\phi \mapsto \phi(a)$ is a bounded linear functional on Φ, and is of norm 1. By the Hahn–Banach Theorem, this extends to a (real valued) bounded linear functional on $C_{\mathbb{R}}(\partial K)$, of norm 1. And so by the real case of the Riesz Representation Theorem

(Theorem 4.1.3), there exists a finite real measure λ_a on ∂K such that $|\lambda_a|(\partial K) = 1$, and

$$\phi(a) = \int_{\partial K} \phi \, \mathrm{d}\lambda_a \tag{27}$$

for all $\phi \in \Phi$. Now note that $\phi \equiv 1$ is in Φ, with an extension that is identically 1 on K°. Hence $\lambda_a(\partial K) = \int_{\partial k} 1 \, \mathrm{d}\lambda_a = 1$. As also $|\lambda_a|(\partial K) = 1$, its negative variation must be zero, and so λ_a is a positive measure.[9]

Now let u_a be the logarithmic potential with respect to λ_a. For $z \in \mathbb{C} \setminus K$, the integrand of $u_a(z)$ is in Φ, with the obvious extension. Hence (27) applies. Extending λ_a by zero to K°, and letting v_a be the logarithmic potential with respect to $\lambda_a - \delta_a$, this gives that $v_a(z) = 0$ for $z \in \mathbb{C} \setminus K$.[10]

Furthermore, for $z_0 \in \partial K$, by Lemma 4.3.7,

$$\begin{aligned}
u_a(z_0) &= \int_{\partial K} \log \left| \frac{1}{z_0 - w} \right| \mathrm{d}\lambda_a(w) \\
&\leqslant \lim_{\substack{z \to z_0 \\ z \in \mathbb{C} \setminus K}} \int_{\partial K} \log \left| \frac{1}{z - w} \right| \mathrm{d}\lambda_a(w) \\
&= \lim_{\substack{z \to z_0 \\ z \in \mathbb{C} \setminus K}} \log \left| \frac{1}{z - a} \right| \\
&= \log \left| \frac{1}{z_0 - a} \right| \\
&< \infty.
\end{aligned}$$

This implies that v_a converges absolutely on ∂K, hence applying Lemma 4.3.2 to K (with Ω equal to $\mathbb{C} \setminus K$), gives that $v_a(z) = 0$ for $z \in \partial K$. So for all $z \in \partial K$,

$$\log \left| \frac{1}{z - a} \right| = \int_{\partial K} \log \left| \frac{1}{z - w} \right| \mathrm{d}\lambda_a(w). \tag{28}$$

(This did not follow from (27) because the integrand is not continuous on ∂K, because $z \in \partial K$.)

Furthermore Lemma 4.3.2 (now on ∂K) assures us that u converges absolutely L-almost everywhere in the plane. Hence by (28), for L-almost all $a \in K^\circ$,

$$\int_{\partial K} \int_{\partial K} \log \left| \frac{1}{z - w} \right| \mathrm{d}\lambda_a(z) \, |\mathrm{d}\alpha(w)| = \int_{\partial K} \log \left| \frac{1}{a - w} \right| |\mathrm{d}\alpha(w)| < \infty.$$

This justifies the use of Fubini's Theorem in the next step. It also shows that λ_a vanishes on the subset of ∂K where u diverges. Combine this with the fact that u is

[9] It is called the *harmonic measure*, and we shall look at it more closely after this result.

[10] We might be concerned about how λ_a and δ_a appear to be 'doing the same job', that is representing evaluation at a. Does this not conflict with the uniqueness part of the Riesz Representation Theorem? Not so, as whilst δ_a represents the evaluation map, λ_a represents the evaluation *of the harmonic extension from the boundary*. Hence the linear functional associated with $\delta_a - \lambda_a$ need only be zero on harmonic functions.

zero where it converges absolutely, and (28), to deduce that for L-almost all $a \in K^\circ$:

$$\begin{aligned}
0 &= \int_{\partial K} u(z)\,\mathrm{d}\lambda_a(z)\\
&= \int_{\partial K}\int_{\partial K} \log\left|\frac{1}{z-w}\right| \mathrm{d}\alpha(w)\,\mathrm{d}\lambda_a(z)\\
&= \int_{\partial K}\int_{\partial K} \log\left|\frac{1}{z-w}\right| \mathrm{d}\lambda_a(z)\,\mathrm{d}\alpha(w)\\
&= \int_{\partial K} \log\left|\frac{1}{a-w}\right| \mathrm{d}\alpha(w)\\
&= u(a).
\end{aligned}$$

Hence u is zero for L-almost all points in K°.

As we also know that u is zero in $\mathbb{C} \setminus K$, and is zero for L-almost all points on ∂K, Lemma 4.3.5 implies that $\alpha \equiv 0$. Hence by the real case of the Riesz Representation Theorem (Theorem 4.1.3), the only bounded linear functional that annihilates $\operatorname{Re} P(\partial K)$ is the zero functional. Thus $\operatorname{Re} P(\partial K)$ being closed implies that $\operatorname{Re} P(\partial K) = C_{\mathbb{R}}(\partial K)$. □

4.4 Harmonic Measures

Consider any $\phi \in C_{\mathbb{R}}(\partial K)$. The Walsh–Lebesgue Theorem gives polynomials whose real parts converge uniformly to ϕ on ∂K. By the maximum principle, and the fact that a uniform limit of harmonic functions is harmonic, they also converge uniformly to some harmonic function in the interior. This harmonic function is then a suitable harmonic extension of ϕ, and so in fact $\Phi = C_{\mathbb{R}}(\partial K)$. Thus the Walsh–Lebesgue Theorem proves existence of a solution to a general version of the Dirichlet problem, in which the complement is connected, and the boundary function is continuous.

This means that when we applied the Hahn–Banach Theorem to define λ_a, we were not actually extending our evaluation functional at all – and so this measure is unique. This motivates the following definition, which obviously coincides with our usage of λ_a above:

Definition 4.4.1. Let $K \subseteq \mathbb{C}$ be compact, and such that $\mathbb{C} \setminus K$ is connected. Let $a \in K^\circ$. Identifying a function $\phi \in C_{\mathbb{R}}(\partial K)$ with its harmonic extension to K°, the map $\phi \mapsto \phi(a)$ is a positive bounded linear functional on $C_{\mathbb{R}}(\partial K)$. So by the Riesz Representation Theorem (Theorem 4.1.4), it has an associated finite positive measure λ_a, called the *harmonic measure*.

Remark 4.4.2. It is immediate from the definition of the harmonic measure that for any $\phi \in C_{\mathbb{R}}(\partial K)$ that $u\colon K^\circ \to \mathbb{R}$ defined by

$$u(z) = \int_{\partial K} \phi(w)\,\mathrm{d}\lambda_z(w)$$

is harmonic.

Definition 4.4.3. Let $K \subseteq \mathbb{C}$. Then the *open components* of K are the connected components of K°.

Lemma 4.4.4. *Let $K \subseteq \mathbb{C}$ be compact, and such that $\mathbb{C} \setminus K$ is connected. Let C be an open component of K. Let $a \in C$. Then λ_a is concentrated on ∂C.*

Proof. For any $\phi \in C_{\mathbb{R}}(\partial K)$, then $\phi \mathbb{1}_{\partial C} \in C_{\mathbb{R}}(\partial C)$. Let λ'_a be the harmonic measure for functions in $C_{\mathbb{R}}(\partial C)$. Extend λ'_a by zero to $\partial K \setminus \partial C$. By the uniqueness of harmonic extensions, ϕ and $\phi \mathbb{1}_{\partial C}$ have the same harmonic extensions to C, and so (identifying ϕ with its harmonic extension),

$$\begin{aligned}\int_{\partial K} \phi \, \mathrm{d}\lambda_a = \phi(a) = (\phi \mathbb{1}_{\partial C})(a) &= \int_{\partial C} \phi \mathbb{1}_{\partial C} \, \mathrm{d}\lambda'_a \\ &= \int_{\partial C} \phi \, \mathrm{d}\lambda'_a \\ &= \int_{\partial K} \phi \, \mathrm{d}\lambda'_a.\end{aligned}$$

That this is true for all $\phi \in C_{\mathbb{R}}(\partial K)$ implies, by the uniqueness part of the real case of the Riesz Representation Theorem (Theorem 4.1.3), that $\lambda_a = \lambda'_a$. The result then follows, as λ'_a is concentrated on ∂C. □

Lemma 4.4.5 (Parseval relation)**.** *Let $K \subseteq \mathbb{C}$ be compact, and such that $\mathbb{C} \setminus K$ is connected. Let $a \in K^\circ$. Let $f \in A(K)$ be such that $f(a) = 0$. Then*

$$\int_{\partial K} (\operatorname{Re} f)^2 \, \mathrm{d}\lambda_a = \int_{\partial K} (\operatorname{Im} f)^2 \, \mathrm{d}\lambda_a.$$

Proof. As $\operatorname{Re}(f^2)$ is harmonic,

$$\begin{aligned}\int_{\partial K} (\operatorname{Re} f)^2 \, \mathrm{d}\lambda_a &= \int_{\partial K} \operatorname{Re}(f^2) \, \mathrm{d}\lambda_a + \int_{\partial K} (\operatorname{Im} f)^2 \, \mathrm{d}\lambda_a \\ &= \operatorname{Re}(f(a)^2) + \int_{\partial K} (\operatorname{Im} f)^2 \, \mathrm{d}\lambda_a \\ &= \int_{\partial K} (\operatorname{Im} f)^2 \, \mathrm{d}\lambda_a.\end{aligned}$$

□

Next we prove a coarse version of Harnack's inequality, which is standard – for example, see [16, Proposition 7.6.1]. See also [8, Lemma 6].

Lemma 4.4.6. *Let $z \in \mathbb{C}$ and $r > 0$. Let $a \in B(z, r)$. Then there exists a fixed constant $c > 0$ such that*

$$u(a) \leqslant cu(z)$$

for all $u \in_{\mathbb{R}} (B(z,r))$ which are non-negative and harmonic.

Proof. Pick $s > 0$ such that $B(a, s) \subseteq B(z, r)$. By the mean value property for harmonic functions (Theorem 4.1.7), with $w = \zeta + \mathrm{i}\eta$,

$$\begin{aligned}u(a) &= \frac{1}{\pi s^2} \iint_{B(a,s)} u(w) \, \mathrm{d}\zeta \, \mathrm{d}\eta \\ &\leqslant \frac{1}{\pi s^2} \iint_{B(z,r)} u(w) \, \mathrm{d}\zeta \, \mathrm{d}\eta \\ &= \frac{r^2}{s^2} u(z).\end{aligned}$$

□

Theorem 4.4.7 (Harnack's Inequality). *Let $\Omega \subseteq \mathbb{C}$ be open and connected. Let $a, b \in \Omega$. Then there exists a fixed constant $c > 0$ such that*

$$u(a) \leqslant cu(b)$$

for all $u \in C_{\mathbb{R}}(\Omega)$ which are non-negative and harmonic.

Proof. As Ω is an open connected subset of the plane, it is path-connected. So let γ be some path between a and b. By compactness of γ we may find points $z_i \in \gamma$ and $r_i > 0$, for $i \in \{1, \ldots, n\}$, such that $B(z_i, r_i) \subseteq \Omega$, $z_0 = b$, $z_n = a$; and $z_{i+1} \in B(z_i, r_i)$ for $i < n$. As u is harmonic in every $B(z_i, r_i)$, by Lemma 4.4.6 there exists constants $c_1, \ldots, c_n$ such that

$$u(a) = u(z_n) \leqslant c_n u(z_{n-1}) \leqslant \ldots \leqslant \left(\prod_{i=1}^{n} c_i\right) u(z_0) = \left(\prod_{i=1}^{n} c_i\right) u(b).$$

□

The next lemma gets its name because it is a specialisation of the notion of 'parts' of the space of multiplicative measures on uniform algebras – see [11, Chapter VI].

Lemma 4.4.8 (Parts lemma). *Let $K \subseteq \mathbb{C}$ be compact, and such that $\mathbb{C} \setminus K$ is connected. Let C be an open component of K, and let $a, b \in C$. Then λ_a and λ_b are mutually absolutely continuous.*

Proof. Let $\phi \in C_{\mathbb{R}}(\partial K)$ be non-negative. Now as the harmonic measure is positive, $u \colon C \to \mathbb{R}$ defined by

$$u(z) = \int_{\partial K} \phi(w) \, \mathrm{d}\lambda_z(w)$$

is non-negative and harmonic in C, see Remark 4.4.2. So apply Harnack's inequality to find constants $c_1, c_2 > 0$ such that

$$c_1 u(a) \leqslant u(b) \leqslant c_2 u(a).$$

Hence

$$c_1 \int_{\partial K} \phi(w) \, \mathrm{d}\lambda_a(w) \leqslant \int_{\partial K} \phi(w) \, \mathrm{d}\lambda_b(w) \leqslant c_2 \int_{\partial K} \phi(w) \, \mathrm{d}\lambda_a(w). \tag{29}$$

To complete the argument, we adapt [21, p. 41]: pick $E \subseteq \partial K$ compact. Fix $\varepsilon > 0$. By regularity[11] of λ_b, there exists $U \subseteq \partial K$ open in ∂K, containing E, such that $\lambda_b(U) < \lambda_b(E) + \varepsilon$. By the definition of the subspace topology[12], we may apply Urysohn's Lemma (Lemma 2.3.5) to find $\phi \in C_{\mathbb{R}}(\partial K)$ such that $\mathbb{1}_E(w) \leqslant \phi(w) \leqslant \mathbb{1}_U(w)$ for $w \in \partial K$. Then by (29),

$$c_1 \lambda_a(E) = c_1 \int_{\partial K} \mathbb{1}_E \lambda_a \leqslant c_1 \int_{\partial K} \phi \lambda_a \leqslant \int_{\partial K} \phi \lambda_b \leqslant \int_{\partial K} \mathbb{1}_U \, \mathrm{d}\lambda_b = \lambda_b(U) < \lambda_b(E) + \varepsilon.$$

Hence $c_1 \lambda_a(E) \leqslant \lambda_b(E)$. Similarly $\lambda_b(E) \leqslant c_2 \lambda_a(E)$. By regularity of λ_a and λ_b, these extend to all measurable $E \subseteq \partial K$, and thus the assertion follows. □

[11] Recall that all finite positive measures on Euclidean space are regular.

[12] That $U = V \cap \partial K$, where $V \subseteq \mathbb{C}$ is open.

Lemma 4.4.9. *Let $K \subseteq \mathbb{C}$ be compact, and such that $\mathbb{C} \setminus K$ is connected. Let C be an open component of K. Let μ be a complex measure on ∂K. Then it has a (Lebesgue) decomposition $\mu = h_C + \sigma_C$ such that $h_C \ll \lambda_a$ and $\sigma_C \perp \lambda_a$ for all $a \in C$.*

Proof. Pick any $b \in C$. Then λ_b is a finite positive measure, so by the Lebesgue Decomposition Theorem (Theorem 4.2.6), we may decompose $\mu = h_C + \sigma_C$ such that $h_C \ll \lambda_b$, $\sigma_C \perp \lambda_b$. Now pick any $a \in C$. Then by the parts lemma, λ_a and λ_b are mutually absolutely continuous, so by parts (iv) and (v) of Lemma 4.2.5, $h_C \ll \lambda_a$, $\sigma_C \perp \lambda_a$. □

Remark 4.4.10. As a consequence of the previous lemma, we shall refer to μ as having a Lebesgue decomposition with respect to the open component C. Similarly we shall write $\nu \perp C$ or $\nu \ll C$.

4.5 Cauchy Transforms

We only have three lemmas in this section. The first is trivial, but used often. The latter two are both of a very similar flavour to previous lemmas relating to logarithmic potentials. As usual we follow [8], here in particular Lemmas 4 and 5, but much of the content of these lemmas is standard – see [11, Chapter II, Section 8], or [26, Lemmas 7.4 and 7.5].

Definition 4.5.1. Let $K \subseteq \mathbb{C}$ be compact. Let μ be a complex measure on K. Define $F\colon \mathbb{C} \to \mathbb{C}$ by

$$F(z) = \int_K \frac{\mathrm{d}\mu(w)}{z-w}.$$

Then F is said to be the *Cauchy transform* of μ.

Lemma 4.5.2. *Let $K \subseteq \mathbb{C}$ be compact. Let μ be a complex measure on K such that*

$$\int_K w^n \, \mathrm{d}\mu(w) = 0$$

for all $n \in \mathbb{N} \cup \{0\}$. Then

$$\int_K h(w) \, \mathrm{d}\mu(w) = 0$$

for all $h \in H(\mathbb{C})$.

Proof. Pick some large disc containing K. Then h will have a Taylor expansion $h(w) = \sum_{n=0}^{\infty} c_n w^n$ in this disc. So by uniform convergence,

$$\begin{aligned}\int_K h(w) \, \mathrm{d}\mu(w) &= \int_K \sum_{n=0}^{\infty} c_n w^n \, \mathrm{d}\mu(w) \\ &= \sum_{n=0}^{\infty} c_n \int_K w^n \, \mathrm{d}\mu(w) \\ &= 0.\end{aligned}$$

□

Lemma 4.5.3. *Let $K \subseteq \mathbb{C}$ be compact, and such that $\mathbb{C} \setminus K$ is connected. Let F be the Cauchy transform of μ, a complex measure on ∂K. Then F converges absolutely L-almost everywhere in $\mathbb{C}$. Furthermore, if $F(z) = 0$ for all $z \in \mathbb{C} \setminus K$, then $F(z) = 0$ for all $z \in \partial K$ at which F converges absolutely.* (*Compare Lemma* 4.3.2.)

Proof. We reason similarly to the start of Lemma 4.3.2. Let $G\colon \mathbb{C} \to \mathbb{C}$ be defined by:

$$G(z) = \int_{\partial K} \frac{|\mathrm{d}\mu(w)|}{|z-w|}.$$

Pick any $R > 0$. Let $S = R + \sup_{w \in \partial K} |w|$. Let $z = x + \mathrm{i}y = w + r\mathrm{e}^{\mathrm{i}\theta}$. Then by applying Fubini's Theorem (justified as our eventual result is finite),

$$\begin{aligned}
\iint_{B(0,R)} G(z)\,\mathrm{d}x\,\mathrm{d}y &= \iint_{B(0,R)} \int_{\partial K} \frac{|\mathrm{d}\mu(w)|}{|z-w|}\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant \int_{\partial K} \iint_{B(w,S)} \frac{1}{|z-w|}\,\mathrm{d}x\,\mathrm{d}y\,|\mathrm{d}\mu(w)| \\
&= \int_{\partial K} \int_0^{2\pi} \int_0^S 1\,\mathrm{d}r\,\mathrm{d}\theta\,|\mathrm{d}\mu(w)| \\
&< \infty.
\end{aligned}$$

Hence $G(z) < \infty$ for L-almost all $z \in B(0,R)$. As R was arbitrary, F converges absolutely L-almost everywhere in $\mathbb{C}$.

For the next part, suppose that $F(z) = 0$ for all $z \in \mathbb{C} \setminus K$, and that F converges absolutely at $z_0 \in \partial K$. We will show that $F(z_0) = 0$. As $1/(z-w)$ may be geometrically expanded for $|z| > \sup_{w \in \partial K} |w|$,

$$\begin{aligned}
0 = F(z) &= \int_{\partial K} \frac{\mathrm{d}\mu(w)}{z-w} \\
&= \int_{\partial K} \sum_{n=0}^{\infty} w^n z^{-n-1}\,\mathrm{d}\mu(w) \\
&= \sum_{n=0}^{\infty} z^{-n-1} \int_{\partial K} w^n\,\mathrm{d}\mu(w),
\end{aligned}$$

commuting sum and integral by uniform convergence of power series. So by uniqueness of Laurent expansions, every coefficient must be zero, giving $\int_{\partial K} w^n\,\mathrm{d}\mu(w) = 0$, and hence that

$$\int_{\partial K} h(w)\,\mathrm{d}\mu(w) = 0 \tag{30}$$

for $h \in H(\mathbb{C})$, by Lemma 4.5.2.

Now $|z_0 - w|$ is a continuous function on ∂K, and so by the Walsh–Lebesgue Theorem (Theorem 4.3.9), there exists a sequence of polynomials (P_n) such that $|\mathrm{Re}(P_n(w)) - |z_0 - w|| \leqslant \frac{1}{2n}$ for $w \in \partial K$. Now let $Q_n = n(P_n - P_n(z_0))$. Then (Q_n) is a sequence of polynomials such that for all n,

$$Q_n(z_0) = 0, \tag{31}$$

$$\mathrm{Re}(Q_n(w)) - n|z_0 - w| \geqslant -1. \tag{32}$$

Then define a sequence of functions $h_n\colon \mathbb{C} \to \mathbb{C}$ by

$$h_n(w) = \frac{1 - \mathrm{e}^{-Q_n(w)}}{z_0 - w},$$

noting by (31) that the (h_n) have removable singularities at z_0, and hence may be defined there. Thus $h_n \in H(\mathbb{C})$. Furthermore (32) implies that (h_n) converges pointwise to $1/(z_0 - w)$ as $n \to \infty$. It also shows that for $w \in \partial K$,

$$|h_n(w)| \leqslant \frac{1 + \mathrm{e}^{-\operatorname{Re}(Q_n(w))}}{|z_0 - w|} \leqslant \frac{1 + e}{|z_0 - w|}.$$

Then F converging absolutely at z_0 means that this is a suitable control function with which to apply the Dominated Convergence Theorem to $\int_{\partial K} h_n(w)\,\mathrm{d}\mu(w)$, which along with (30) gives

$$0 = \int_{\partial K} h_n(w)\,\mathrm{d}\mu(w) \longrightarrow \int_{\partial K} \frac{1}{z_0 - w}\,\mathrm{d}\mu(w) = F(z_0).$$

□

Lemma 4.5.4. *Let $K \subseteq \mathbb{C}$ be compact, and such that $\mathbb{C} \setminus K$ is connected. Let F be the Cauchy transform of μ, a complex measure on ∂K. If $F(z) = 0$ for L-almost all $z \in \mathbb{C}$, then $\mu \equiv 0$.* (*Compare Lemma* 4.3.5.)

Proof. Pick any $\Phi \in C^1(\partial K)$. Then $\Phi \in C^1_c(\mathbb{C})$, so applying Lemma 2.3.1 and Fubini's Theorem yields, with $w = \zeta + \mathrm{i}\eta$,

$$\begin{aligned}
\int_{\partial K} \Phi(z)\,\mathrm{d}\mu(z) &= -\frac{1}{\pi}\int_{\partial K}\left[\iint_{\mathbb{C}} \frac{(\partial_{\bar z}\Phi)(w)}{w - z}\,\mathrm{d}\zeta\,\mathrm{d}\eta\right]\mathrm{d}\mu(z) \\
&= -\frac{1}{\pi}\iint_{\mathbb{C}} (\partial_{\bar z}\Phi)(w)\left[\int_{\partial K} \frac{1}{w - z}\,\mathrm{d}\mu(z)\right]\mathrm{d}\zeta\,\mathrm{d}\eta \\
&= -\frac{1}{\pi}\iint_{\mathbb{C}} (\partial_{\bar z}\Phi)(w) F(w)\,\mathrm{d}\zeta\,\mathrm{d}\eta \\
&= 0.
\end{aligned}$$

Now $C^1(\partial K)$ is dense in $C(\partial K)$, hence the bounded linear map $f \mapsto \int_{\partial K} f(z)\,\mathrm{d}\alpha(z)$ on $C(K)$ is the zero map. Thus the zero measure may be associated with this map and so, as the Riesz Representation Theorem (Theorem 4.1.4) asserts that associated measures are unique, $\mu \equiv 0$. □

Remark 4.5.5. Similar to Remark 4.3.6, the previous lemma shows that a measure is determined by its Cauchy transform. Additionally, the previous two lemmas are actually sufficient to prove Mergelyan's Theorem in the case of no interior points, when $\partial K = K$: suppose that μ is a complex measure annihilating $P(K)$. Then by the pole-pushing lemma (Lemma 2.3.3), its Cauchy transform is zero in $\mathbb{C} \setminus K$. Hence Lemma 4.5.3 shows that F is zero L-almost everywhere in the plane, and so Lemma 4.5.4 shows that $\mu \equiv 0$.

4.6 Mergelyan's Theorem

We are now ready to tackle Mergelyan's Theorem itself! Strong links are visible between the approach in this section and the abstract approach taken in [14], which assumes results analogous to the results of our previous sections. As usual we adapt [8], here in particular Lemmas 7 and 8. We begin by presenting an abstract form of the F. and M. Riesz Theorem, also see [13] or [11, Chapter II, Theorem 7.6].

Lemma 4.6.1. *Let $K \subseteq \mathbb{C}$ be compact, and such that $\mathbb{C} \setminus K$ is connected. Let $a \in K^\circ$. Let $g_n \in A(K)$ for $n \in \mathbb{N}$. Suppose that*

$$\sum_{n=1}^{\infty} \int_{\partial K} |g_n|^2 \, \mathrm{d}\lambda_a < \infty.$$

Then $g_n \to 0$ $[\lambda_a]$.

Proof. Treating the sum as an integral with respect to the counting measure, Fubini's Theorem allows us to interchange summation and integration. Hence for λ_a-almost all $z \in \partial K$,

$$\sum_{n=1}^{\infty} |g_n(z)|^2 < \infty.$$

Hence $g_n \to 0$ $[\lambda_a]$. □

Theorem 4.6.2 (F. and M. Riesz Theorem)**.** *Let $K \subseteq \mathbb{C}$ be compact, and such that $\mathbb{C} \setminus K$ is connected. Let C be an open component of K. Let μ be a complex measure on ∂K annihilating $P(\partial K)$. Let $\mu = h_C + \sigma_C$ be the Lebesgue decomposition of μ with respect to C. Then h_C annihilates $P(\partial C)$. (And hence also annihilates $P(\partial K)$, as we may view it as a subset of $P(\partial C)$.) Furthermore for $a \in C$,*

$$\int_{\partial K} \frac{\mathrm{d}h_C(z)}{a - z} = \int_{\partial K} \frac{\mathrm{d}\mu(z)}{a - z}.$$

Proof. Fix $a \in C$. We begin by defining a sequence of polynomials (P_n), and showing that they have particular properties.

Now σ_C and λ_a are mutually singular, so let them be concentrated on disjoint $Y, Z \subseteq \partial K$ respectively. Furthermore σ_C is a complex measure and so $|\sigma_C|$ is finite, hence there exist $Y_n \subseteq Y$ closed, increasing, and such that $\sigma_C(Y \setminus Y_n) \to 0$ as $n \to \infty$. Similarly there exist $Z_n \subseteq Z$ closed and such that $\lambda_a(Z \setminus Z_n) \to 0$ as $n \to \infty$. By relabelling if necessary, we may assume that in fact $\lambda_a(Z \setminus Z_n) \leqslant 2^{-4n-1}/9$.

Let $f_n \colon Y_n \cup Z_n \to \mathbb{R}$ be defined by

$$f_n(z) = \begin{cases} 1 + 2^n & \text{if } z \in Y_n, \\ 2^{-n} & \text{if } z \in Z_n. \end{cases}$$

Now Y, Z disjoint implies that Y_n, Z_n are disjoint. As they are closed, f_n is continuous. So apply the real case of the Tietze Extension Theorem (Theorem 2.3.6) to extend f_n to an element of $C_c(\mathbb{C})$; then restrict to ∂K to produce an element of $C_{\mathbb{R}}(\partial K)$, which we shall denote again by f_n.

The Walsh–Lebesgue Theorem (Theorem 4.3.9) now allows f_n to be uniformly approximated on ∂K by the real parts of polynomials. Pick polynomials P_n such that

$$|f_n(z) - \mathrm{Re}(P_n(z))| < 2^{-n-1} \tag{33}$$

for all $z \in \partial K$. Then by (33),

$$\begin{aligned}\int_{Z_n} (\mathrm{Re}\, P_n(z))^2 \,\mathrm{d}\lambda_a(z) &\leqslant \lambda_a(Z_n) \cdot \sup_{z\in Z_n} (\mathrm{Re}\, P_n(z))^2 \\ &\leqslant \lambda_a(\partial K) \cdot \sup_{z\in Z_n} (f_n(z) - 2^{-n-1})^2 \\ &= (2^{-n} - 2^{-n-1})^2 \\ &\leqslant 2^{-2n-1}.\end{aligned}$$

As the Tietze extension f_n has the same supremum as the original f_n, then (33) also gives that

$$\begin{aligned}\int_{Z\setminus Z_n} (\mathrm{Re}\, P_n(z))^2 \,\mathrm{d}\lambda_a(z) &\leqslant \lambda_a(Z\setminus Z_n) \cdot \sup_{z\in Z\setminus Z_n} (\mathrm{Re}\, P_n(z))^2 \\ &\leqslant \lambda_a(Z\setminus Z_n) \cdot \sup_{z\in \partial K} (f_n(z) - 2^{-n-1})^2 \\ &= \lambda_a(Z\setminus Z_n) \cdot (1 + 2^n - 2^{-n-1})^2 \\ &\leqslant 9 \cdot \lambda_a(Z\setminus Z_n) \cdot 2^{2n} \\ &\leqslant 2^{-2n-1}.\end{aligned}$$

So combine these last two to deduce that

$$\int_{\partial K} (\mathrm{Re}\, P_n(z))^2 \,\mathrm{d}\lambda_a(z) = \int_Z (\mathrm{Re}\, P_n(z))^2 \,\mathrm{d}\lambda_a(z) \leqslant 2^{-2n}. \tag{34}$$

This then implies by the Cauchy–Schwarz inequality that

$$|\mathrm{Re}(P_n(a))| = \left| \int_{\partial K} \mathrm{Re}(P_n) \,\mathrm{d}\lambda_a \right| \leqslant 2^{-n}. \tag{35}$$

Additionally, for $z \in Y_n$,

$$\mathrm{Re}(P_n(z)) \geqslant f_n(z) - 2^{-n-1} = 1 + 2^n - 2^{-n-1} \geqslant 2^n. \tag{36}$$

Now let $Q_n = P_n - P_n(a)$. Then by our Parseval relation (Lemma 4.4.5), and (34), and (35),

$$\sum_{n=1}^{\infty} \int_{\partial K} |Q_n|^2 \,\mathrm{d}\lambda_a = 2 \sum_{n=1}^{\infty} \int_{\partial K} (\mathrm{Re}\, Q_n)^2 \,\mathrm{d}\lambda_a < \infty.$$

So by Lemma 4.6.1, $Q_n \to 0$ $[\lambda_a]$. As $h_C \ll \lambda_a$, then $Q_n \to 0$ $[h_C]$. Furthermore (Y_n) increasing and (36) imply that $\mathrm{e}^{-Q_n} \to 0$ $[\sigma_C]$.

Now (33), (35), and the fact that the Tietze extension f_n has the same infimum as the original f_n imply that $\mathrm{Re}\, Q_n$ is bounded above on ∂K independently of n, and hence $\left|\mathrm{e}^{-Q_n}\right| = \mathrm{e}^{-\operatorname{Re} Q_n}$ is bounded above on ∂K independently of n. So by the Dominated Convergence Theorem, for any $m \in \mathbb{N} \cup \{0\}$,

$$0 = \int_{\partial K} z^m \mathrm{e}^{-Q_n(z)} \, \mathrm{d}\mu(z) \longrightarrow \int_{\partial K} z^m \, \mathrm{d}h_C = \int_{\partial C} z^m \, \mathrm{d}h_C$$

as $n \to \infty$. The first equality follows by Lemma 4.5.2, as the integrand is entire. The last equality follows because $h_C \ll C$, so Lemma 4.4.4 implies that h_C is concentrated on C.

Hence h_C annihilates $P(\partial C)$.

For the last assertion, observe that

$$\int_{\partial K} \frac{\mathrm{d}\mu(z)}{a - z} = \int_{\partial K} \frac{1 - \mathrm{e}^{-Q_n(z)}}{a - z} \, \mathrm{d}\mu(z) + \int_{\partial K} \frac{\mathrm{e}^{-Q_n(z)}}{a - z} \, \mathrm{d}\mu(z).$$

As $Q_n(a) = 0$, the integrand of the first integral has a removable singularity at a. So it is entire, and so by Lemma 4.5.2 this integral must be zero. Then taking $n \to \infty$ gives that

$$\int_{\partial K} \frac{\mathrm{d}\mu(z)}{a - z} = \int_{\partial K} \frac{\mathrm{d}h_C(z)}{a - z}$$

by the Dominated Convergence Theorem. □

Lemma 4.6.3. *Let $K \subseteq \mathbb{C}$ be compact, and such that $\mathbb{C} \setminus K$ is connected. Let C be an open component of K. Let μ be a complex measure on ∂K annihilating $P(\partial K)$. Let $\mu = h_C + \sigma_C$ be a Lebesgue decomposition of μ with respect to C. Let $f \in A(K)$. Then $\int_{\partial K} f \, \mathrm{d}h_C = 0$.*

Proof. Let $M = 2 \sup_{z \in K} |f(z)|$. Let $g(z) = \log(f(z) + M)$, taking the principal branch of log. By the Walsh–Lebesgue Theorem (Theorem 4.3.9), choose polynomials (P_n) such that for all $z \in \partial K$,

$$|\operatorname{Re}(g(z) - P_n(z))| \leqslant 2^{-n-1}.$$

As g is holomorphic, $\operatorname{Re}(g(z) - P_n(z))$ is harmonic, hence

$$|\operatorname{Re}(g(z) - P_n(z))| \leqslant 2^{-n-1} \tag{37}$$

for all $z \in K$, by the maximum principle.

Fix $a \in C$. Let $Q_n = P_n + g(a) - P_n(a)$. Then by (37), for all $z \in \partial K$,

$$|\operatorname{Re}(g(z) - Q_n(z))| \leqslant 2^{-n}, \tag{38}$$

which together with our Parseval relation (Lemma 4.4.5), applied to $g - Q_n$, gives

$$\sum_{n=1}^{\infty} \int_{\partial K} |g - Q_n|^2 \, \mathrm{d}\lambda_a = 2 \sum_{n=1}^{\infty} \int_{\partial K} (\operatorname{Re}(g - Q_n))^2 \, \mathrm{d}\lambda_a < \infty.$$

So by Lemma 4.6.1, $g - Q_n \to 0$ $[h_C]$. Now (38) implies that $\left|\mathrm{e}^{Q_n}\right| = \mathrm{e}^{\mathrm{Re}\, Q_n}$ is bounded above and below on ∂K independently of n. So by the Dominated Convergence Theorem,

$$\begin{aligned}\int_{\partial K} f \,\mathrm{d}h_C &= \int_{\partial K} (f + M) \,\mathrm{d}h_C \\ &= \int_{\partial K} \mathrm{e}^{g(z)} \,\mathrm{d}h_C(z) \\ &= \lim_{n\to\infty} \int_{\partial K} \mathrm{e}^{Q_n}(z) \,\mathrm{d}h_C(z) \\ &= 0.\end{aligned}$$

The last equality follows by Lemma 4.5.2 as the integrand is entire, along with the fact that h_C annihilates $P(\partial K)$, by the F. and M. Riesz Theorem. □

Theorem 4.6.4 (Mergelyan's Theorem)**.** *Let $K \subseteq \mathbb{C}$ be compact, and such that $\mathbb{C} \setminus K$ is connected. Then $f \in P(K)$ if and only if $f \in A(K)$.*

Proof. The 'only if' direction is trivial and is the same as the complex analytic proof, which is Theorem 3.3.4. Conversely suppose that $f \in A(K)$. Then identifying f with its restriction, $f \in C(\partial K)$. Let μ be any complex measure annihilating $P(\partial K)$.

Let $C_1, \dots C_m, \dots$ be the open components of K. (There are at most countably many, as an open set is the union of at most countably many open balls.) We shall assume without loss of generality that there are an infinite number of them, as the finite case is strictly simpler, with the argument following the same lines. Let $\mu = h_{C_1} + \sigma_{C_1}$ be the Lebesgue decomposition of μ with respect to C_1. Now repeat; let $\sigma_{C_1} = h_{C_2} + \sigma_{C_2}$ be the Lebesgue decomposition of σ_{C_1} with respect to C_2. Proceeding in this fashion we obtain two sequences of measures, (h_{C_m}) and (σ_{C_m}).

Fix $i, j \in \mathbb{N}$, such that $i < j$. As $\sigma_{C_i} \perp C_i$, and Lebesgue decompositions are concentration preserving (see Theorem 4.2.6), $h_{C_j} \perp C_i$ and $\sigma_{C_j} \perp C_i$ also. Hence as $h_{C_i} \ll C_i$, Lemma 4.2.5 part (v) implies that $h_{C_j} \perp h_{C_i}$ and $\sigma_{C_j} \perp h_{C_i}$. And so Lemma 4.2.5 part (ii) shows that any finite sum (over j) of h_{C_j} and σ_{C_j} will be mutually singular with h_{C_i}. So by Lemma 4.2.7, for all $k \in \mathbb{N}$,

$$\|\mu\| = \|\sigma_{C_k}\| + \sum_{m=1}^{k} \|h_{C_m}\|.$$

Thus in particular $\sum_{m=1}^{k} \|h_{C_m}\| \leqslant \|\mu\|$, and so $\sum_{m=1}^{\infty} h_{C_m}$ converges absolutely. We recall that the space of complex measures equipped with the $\|\cdot\|$ norm is complete, by Remark 4.1.5, and so in fact this sum converges in this norm. So let $\sigma = \mu - \sum_{m=1}^{\infty} h_{C_m}$.

Next we examine σ.

First fix $z \in \mathbb{C} \setminus K$. Then by the pole-pushing lemma (Lemma 2.3.3), $1/(z - w) \in P(K)$ as a function of w. Hence also $1/(z - w) \in P(\partial K)$. As μ annihilates $P(\partial K)$, then by the F. and M. Riesz Theorem (Theorem 4.6.2), so does h_{C_m}, for all m. Hence σ does also.[13] So in particular,

$$\int_{\partial K} \frac{\mathrm{d}\sigma(w)}{z - w} = 0.$$

[13]By continuity of evaluation on the dual space.

Next, fix any $j \in \mathbb{N}$, and fix $z \in C_j$. Now with C_j in place of K, then $1/(z-w) \in P(\partial C_i)$ for all $i \neq j$. Hence

$$\int_{\partial K} \frac{\mathrm{d}\sigma(w)}{z-w} = 0$$

as the integrals against h_{C_i}, for $i \neq j$, are zero by the first part of the F. and M. Riesz Theorem (Theorem 4.6.2), and the integrals against μ and h_{C_j} cancel, by the second part of the F. and M. Riesz Theorem.

To summarise: we have shown for all $z \in (\mathbb{C} \setminus K) \cup \bigcup_{m=1}^{\infty} C_m = (\mathbb{C} \setminus K) \cup K^\circ$ that $\int_{\partial K} \mathrm{d}\sigma(w)/(z-w) = 0$. Furthermore, our second Cauchy transform lemma (Lemma 4.5.3) implies that the integral is zero for L-almost all $z \in \partial K$. Hence $\int_{\partial K} \mathrm{d}\sigma(w)/(z-w) = 0$ for L-almost all $z \in \mathbb{C}$, and so the third Cauchy transform lemma (Lemma 4.5.4) implies that $\sigma \equiv 0$.[14]

Hence $\mu = \sum_{m=1}^{\infty} h_{C_m}$, so Lemma 4.6.3 implies that

$$\int_{\partial K} f \,\mathrm{d}\mu = \sum_{m=1}^{\infty} \int_{\partial K} f \,\mathrm{d}h_{C_m} = 0,$$

commuting sum and integral by continuity of evaluation on the dual space. That this is true for all measures annihilating $P(\partial K)$ shows by the Riesz Representation Theorem (Theorem 4.1.4) that all bounded linear functions which vanish on $P(\partial K)$ also vanish at f. So $P(\partial K)$ closed implies that $f \in P(\partial K)$. Hence by the maximum principle and the uniqueness of harmonic extensions, $f \in P(K)$. $\square$

5 Rational Approximation

We have mentioned previously that functional analytically, Mergelyan's Theorem is but the tip of a very large iceberg – something which is not necessarily clear when forging a direct path towards our goal. So here we exhibit some of the fauna and flora of function algebras, which Mergelyan's Theorem lives alongside. We previously touched on the fact that many of the elements of our functional analytic proof of Mergelyan's Theorem are simply special cases of far more general results (over Dirichlet algebras, logmodular algebras, ...), and this more general context remains its natural home. A systematic treatment in this broader framework can be found in [11], with further reading in [9], [2] or [28]. An introduction to the topic can be found in [7].

Here we state some of those results which relate closely to Mergelyan's Theorem, and omit nearly all the proofs. In particular, we present some theorems concerning rational approximation, which is the natural generalisation from Mergelyan's Theorem. However, the general question of rational approximation remains open; there is no complete analogue of Mergelyan's Theorem. Many further results of this type exist, see our previous references. (In particular we do not cover the notion of *peak points* here.)

Definition 5.1. A *rational function* is one which is the quotient of two polynomials.

[14] This is a special case of *Wilken's Theorem*, see [11, Chapter II, Theorem 8.5], or [12, Lemma 3.5], who cites [27].

Definition 5.2. The *complementary components* of $K \subseteq \mathbb{C}$ are the connected components of its complement.

Definition 5.3. Let $K \subseteq \mathbb{C}$ be compact. Then let $R(K)$ denote the set of all functions $f\colon K \to \mathbb{C}$ which are uniform limits on K of rational functions in z whose poles lie outside K.

It is clear that in general,

$$P(K) \subseteq R(K) \subseteq A(K) \subseteq C(K).$$

The fundamental question we have been investigating is when any of these inclusions become equalities. The last one is easy: $A(K) = C(K)$ if and only if K has no interior. Mergelyan's Theorem gives $P(K) = R(K) = A(K)$ in the case of K having connected complement. And certainly $P(K) \subsetneq R(K)$ when the complement of K is disconnected: let α be in one of the bounded complementary components of K. Then $1/(z-\alpha)$, as a function of z, is in $R(K)$, but cannot be in $P(K)$, as it is not holomorphic at α (if it was in $P(K)$ the maximum principle would imply holomorphicity in all bounded complementary components). Conversely $P(K) = R(K)$ when the complement of K is connected: this follows from the pole-pushing lemma, see Remark 2.3.4. Hence the only remaining question is to characterise $R(K)$, and in particular when $R(K) = A(K)$, in the case of K having disconnected complement.[15]

We begin with a generalisation of Mergelyan's Theorem, see [12] or [2, Chapter X, Theorem 8.4]:

Theorem 5.4 (Mergelyan's Theorem)**.** *Let $K \subseteq \mathbb{C}$ be compact, with at most finitely many complementary components. Then $f \in R(K)$ if and only if $f \in A(K)$.*

This gives a complete answer in the case of finitely many complementary components. What about infinitely many complementary components? Sadly, Mergelyan's Theorem does not generalise, as the 'Swiss cheese' example demonstrates; we follow [28, Example 9.6], see also [20, No. 2.4] or [11, Chapter II, Section 1].

Example 5.5 (Mergelyan's Swiss Cheese)**.** Let $B = B(0,1)$, and enumerate the rational points of B as $\{q_n\}$. We shall pick numbers $r_n > 0$: for these, define $B_n = B(q_n, r_n)$. Begin by picking $r_1 \in (0, \frac{1}{2})$ such that $\overline{B_1} \subseteq B$.

Now for each n in turn, check if $q_n \in \bigcup_{i=1}^{n-1} \overline{B_i}$. If it is, skip it. If it is not, pick $r_n \in (0, 2^{-n})$ such that $\overline{B_n}$ is disjoint from $\bigcup_{i=1}^{n-1} \overline{B_i}$. Proceeding in this fashion, we produce a sequence of open discs $\{B_n\}$, which we have relabelled to remove all those q_n we skipped. See Figure 2. Then:

(i) For all n, $\overline{B_n} \subseteq B$.

(ii) For $n \neq m$, $\overline{B_n} \cap \overline{B_m} = \varnothing$.

(iii) $\sum_{n=1}^{\infty} r_n < \infty$.

(iv) $K = \overline{B} \setminus \bigcup_{n=1}^{\infty} B_n$ has empty interior.

[15] Observe how we have now transitioned to considering the properties of $K \subseteq \mathbb{C}$, rather than $f \in P(K)$.

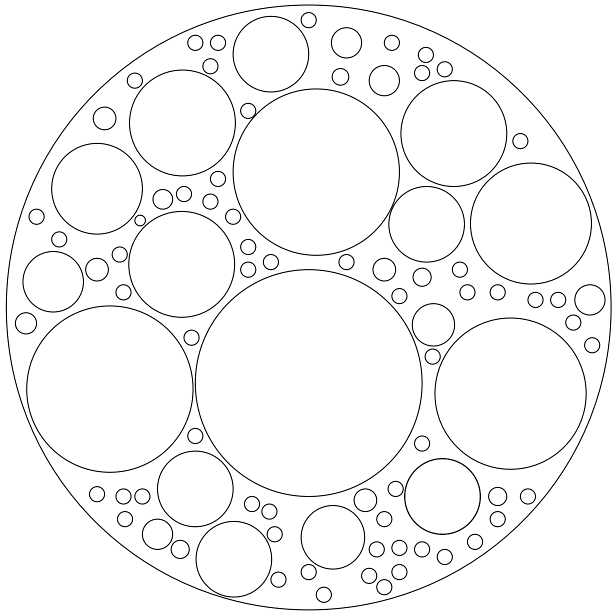

Figure 2: Mergelyan's Swiss Cheese

Now let μ be a measure equal to $\mathrm{d}z$ on ∂B, and equal to $-\mathrm{d}z$ on $\bigcup_{n=1}^{\infty} \partial B_n$. By (i) this is well defined; by (iii), μ is a finite measure. It is certainly a nonzero measure. Let Λ be the bounded linear functional on $C(K)$ associated with μ.

Then Λ is nonzero and annihilates $R(K)$; this is because the residue from a pole in B_n appears once in the integral around ∂B, and once negatively in the integral around ∂B_n, and thus cancels out. Hence $R(K) \neq C(K)$ ($= A(K)$ as K has no interior).

In fact, it is possible to find even worse cases where rational approximation fails. [11, Chapter VIII, Section 9] provides examples of:

(i) A compact set $K \subseteq \mathbb{C}$ whose interior is connected and dense in K, and whose boundary has zero Lebesgue measure, but for which $R(K) \neq A(K)$. (Compare with the Hartogs–Rosenthal Theorem (Theorem 5.7).)

(ii) A compact set $K \subseteq \mathbb{C}$ whose interior is simply connected and dense in K, but for which $R(K) \neq A(K)$.

(iii) A compact set $K \subseteq \mathbb{C}$ whose interior consists of two connected components C_1, C_2, both simply connected, such that $K = \overline{C_1} \cup \overline{C_2}$, and such that $R(C_j) = A(C_j)$ for $j \in \{1, 2\}$, but for which nonetheless $R(K) \neq A(K)$.

Despite this, Mergelyan's Theorem can be extended to infinitely many complementary components – provided that the diameters of those components are bounded away from zero [11, Chapter II, Theorem 10.4]. Furthermore, Runge's Theorem does directly generalise to infinitely many complementary components: indeed, it is usually stated in this form [21, Theorem 13.6].

Theorem 5.6 (Runge's Theorem)**.** *Let $K \subseteq \mathbb{C}$ be compact. Suppose that Ω is an open set containing K. Suppose also that $f \in H(\Omega)$. Fix $\varepsilon > 0$. Let P be a set consisting of precisely one point in each complementary component of K. Then there exists a rational function $r(z)$ with poles only in P such that*

$$\sup\{|f(z) - r(z)| : z \in K\} < \varepsilon.$$

Hence in particular, $f \in R(K)$.

We can recover our earlier version of Runge's Theorem by assuming that $\widehat{\mathbb{C}} \setminus K$ is connected, and letting $P = \{\infty\}$.

Compare the proof of the next theorem with the discussion in Remark 4.5.5. By arguing similarly, and invoking the second Cauchy transform lemma (Lemma 4.5.3), it is possible to slightly generalise this theorem to those compact $K \subseteq \mathbb{C}$ which lie on the boundary of their unbounded complementary component, a notion which will be familiar to readers of [14]. We follow [26, Theorem 7.8], alternatively [11, Chapter II, Corollary 8.4].

Theorem 5.7 (Hartogs–Rosenthal Theorem)**.** *Let $K \subseteq \mathbb{C}$ be compact and with Lebesgue measure zero. Then $R(K) = C(K)$.*

Proof. Suppose for contradiction that $R(K) \neq C(K)$. Then by the Riesz Representation Theorem (Theorem 4.1.4), there exists some nonzero measure μ annihilating $R(K)$. Hence in particular, for all $z \in \mathbb{C} \setminus K$,

$$\int_K \frac{\mathrm{d}\mu(w)}{z - w} = 0.$$

As K has Lebesgue measure zero, the integral is zero L-almost everywhere. Hence by Lemma 4.3.5, $\mu \equiv 0$. This is a contradiction, hence $R(K) = C(K)$. □

Remark 5.8. The previous theorem, combined with the 'Swiss cheese' example, provides a roundabout way of showing a topological oddity: a compact set whose boundary has positive Lebesgue measure. The 'ε-Cantor Set' [1, pp. 140–141] is another such example: it is nowhere dense, yet has positive Lebesgue measure.

To progress further with the problem of rational approximation, we need some additional definitions.

Definition 5.9. Let $K \subseteq \mathbb{C}$ be compact. A subset $X \subseteq C(K)$ is an *algebra* if for all $f, g \in X$ and $\lambda \in \mathbb{C}$ that $f + g$, fg, $\lambda f \in X$ also.

Definition 5.10. Let $K \subseteq \mathbb{C}$ be compact. A subset $X \subseteq C(K)$ is a *Dirichlet algebra* if it is an algebra such that

(i) X is closed in $C(K)$;

(ii) For all $x, y \in K$ such that $x \neq y$, there exists $f \in X$ such that $f(x) \neq f(y)$ (X is said to *separate points*);

(iii) X contains every constant function;

(iv) every $f \in C_{\mathbb{R}}(K)$ is the uniform limit of the real parts of functions in X.

For instance, the Walsh–Lebesgue Theorem (Theorem 4.3.9) implies that $P(\partial K)$ is a Dirichlet algebra, provided $K \subseteq \mathbb{C}$ compact with connected complement.

Now, see [11, Chapter II, Corollary 9.3],

Theorem 5.11. *Let $K \subseteq \mathbb{C}$ be compact. If $R(K)$ is a Dirichlet algebra, then $R(K) = A(K)$.*

Extending these notions provides a route to proving Mergelyan's Theorem: this is what is done in [14, Theorem 2], which was historically one of the stepping stones towards developing the functional analytic proof we provide here.

Having seen these previous results – and the lack of a complete answer for rational approximation – we finish with a theorem that might come as a surprise, by seemingly providing a complete characterisation of rational approximation. In practice, though, it simply translates one difficult problem into another. Its presentation here is brief, and only to give a glimpse of the depth of the further theory. First some definitions. We use [11, Chapter VIII, Theorem 8.2], see also [11, Chapter VIII, Theorem 5.1] and [28, Theorem 12.1].

Definition 5.12. Let $K \subseteq \mathbb{C}$. Let $\mathscr{AC}(K)$ denote the set of functions $f \in C(\widehat{\mathbb{C}})$ which are analytic off some compact subset of K, such that $f(\infty)=0$ and $\sup_{z \in \widehat{\mathbb{C}} \setminus K} |f(z)| \leqslant 1$.

A function which is analytic at ∞ has a Laurent expansion about ∞ of the form $f(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots$. Then we define $f'(\infty) = a_1 = \lim_{z \to \infty} z(f(z) - f(\infty))$, which need not equal $\lim_{z \to \infty} f'(z)$.

Definition 5.13. The *continuous analytic capacity* of $K \subseteq \mathbb{C}$ is defined to be

$$\alpha(K) = \sup\{|f'(\infty)| \,:\, f \in \mathscr{AC}(K)\}.$$

Theorem 5.14 (Vitushkin's Theorem)**.** *Let $K \subseteq \mathbb{C}$ be compact. Then the following are equivalent:*

(i) *$R(K) = A(K)$.*

(ii) *For all $U \subseteq \mathbb{C}$ open and bounded, $\alpha(U \setminus K) = \alpha(U \setminus K^\circ)$.*

(iii) *For all $z \in \mathbb{C}$, all $\delta > 0$, and all $r > 1$,*

$$\alpha(B(z,\delta) \setminus K^\circ) \leqslant \alpha(B(z, r\delta) \setminus K).$$

(iv) *There exists $r \geqslant 1$ and $c > 0$ such that for all $z \in \mathbb{C}$ and all $\delta > 0$,*

$$\alpha(B(z,\delta) \setminus K^\circ) \leqslant c\alpha(B(z, r\delta) \setminus K).$$

(v) *For all $z \in \partial K$, there exists $r \geqslant 1$ such that*

$$\limsup_{\delta \to 0} \frac{\alpha(B(z,\delta) \setminus K^\circ)}{\alpha(B(z, r\delta) \setminus K)} < \infty.$$

References

[1] C. D. Aliprantis and O. Burkinshaw. *Principles of Real Analysis*. Academic Press, San Diego, third edition, 1998.

[2] K Barbey and H König. *Abstract Analytic Function Theory and Hardy Algebras*, volume 593. Springer-Verlag, 1969.

[3] E. Bishop. Measures orthogonal to polynomials. *Proc. Natl. Acad. Sci.*, 44(3):278–280, 1958.

[4] E. Bishop. The structure of certain measures. *Duke Math. J.*, 25(2):283–289, 1958.

[5] E. Bishop. A minimal boundary for function algebras. *Pacific J. Math.*, 9(3):629–642, 1959.

[6] E. Bishop. Boundary measures of analytic differentials. *Duke Math. J.*, 27(3):331–340, 1960.

[7] A. Browder. *Introduction to Function Algebras*. W. A. Benjamin, Inc., New York, 1969.

[8] L. Carleson. Mergelyan's theorem on uniform polynomial approximation. *Math. Scand.*, 15:167–175, 1964.

[9] A. M. Davie. Real annihilating measures for R(K). *J. Functional Analysis*, 6(3):357–386, 1970.

[10] R. Engelking. *General Topology*. Heldermann Verlag, Berlin, revised and completed edition, 1989.

[11] T. W. Gamelin. *Uniform Algebras*. Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1950.

[12] J. B. Garnett. On a theorem of Mergelyan. *Pacific J. Math.*, 26(3):461–467, 1968.

[13] I. Glicksberg. The abstract F. and M. Riesz theorem. *J. Functional Analysis*, 1(1):109–122, 1967.

[14] I. Glicksberg and J. Wermer. Measures orthogonal to a Dirichlet algebra. *Duke Math. J.*, 30(4):661–666, 1963.

[15] B. J. Green. On Runge's Theorem. (Unpublished notes, available at `https://people.maths.ox.ac.uk/greenbj/papers/runge.pdf`).

[16] R. E Greene and S. G. Krantz. *Function Theory of One Complex Variable*. American Mathematical Society, Providence, R.I., third edition, 2006.

[17] P. R. Halmos. *Measure Theory*. D. Van Nostrand Company, Inc., New York, first edition, 1950.

[18] F. Hausdorff. Über halbstetige Funktionen und deren Verallgemeinerung. *Math. Z.*, 5:292–309, 1919. (German).

[19] P. D. Lax. *Functional Analysis.* Wiley-Interscience, New York, 2002.

[20] S. N. Mergelyan. Uniform approximations to functions of a complex variable (Russian). *Uspehi Matem. Nauk (N.S.)*, 7(2 (48)):31–122, 1952.
English translation in *Amer. Math. Soc. Translation*, 101:99, 1954, reprinted in Translations, volume 3 of 1: *Series and Approximation*, pp. 295–391. American Mathematical Society, Providence, R.I, 1962.

[21] W. Rudin. *Real and Complex Analysis.* McGraw-Hill Book Company, third edition, 1987.

[22] E. B. Saff. Logarithmic potential theory with applications to approximation theory. *Surv. Approx. Theory*, 5:165–200, 2010.

[23] E. B. Saff and V. Totik. *Logarithmic potentials with external fields.* Springer, 1997.

[24] J. L. Walsh. Über die Entwicklung einer harmonischen Funktion nach harmonischen Polynomen. *J. Reine Angew. Math.*, 159:197–209, 1928. (German).

[25] J. L. Walsh. The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions. *Bull. Amer. Math. Soc.*, 35(4):499–544, 1929.

[26] J. Wermer. Banach algebras and analytic functions. *Advances in Math.*, 1(1):51–102, 1961.

[27] D. R. Wilken. Lebesgue measure for parts of R(X). *Proc. Amer. Math. Soc.*, 18(3):508–512, 1967.

[28] L. Zalcman. *Analytic Capacity and Rational Approximation*, volume 50. Springer, 1968.