Limits of Fisher–KPP equations, branching Brownian motion and a spatial Λ-Fleming–Viot model for population expansion

Patrick Kidger

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Abstract

We introduce a possible spatial Λ-Fleming–Viot model for modelling population expansion, and prove that under various scaling limits it converges to the Fisher–KPP equation, the heat equation, or a particular parabolic equation: the latter two thus rule out certain scaling limits as unphysical. Our proof will exploit a duality with a system of branching and coalescing random walkers which approximate branching Brownian motion.

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1 Introduction

The spatial Λ-Fleming–Viot model provides a way to model genetic evolution in a spatial continuum, and has seen great success in population genetics in modelling the interaction between two different genotypes, see for example [EFP17], [BEV13] or [EVY14]. The question arises as to whether it can be applied to modelling population expansion, in which a species expands uncontested into a new environment; we might hope to model this scenario in a similar manner to genic selection, in which a great advantage is given to one genotype (representing the population) over its rival (representing empty space).

We begin by introducing our spatial Λ-Fleming–Viot model for expansion, and prove that it exhibits a duality with a branching and coalescing jump process. Under appropriate scaling limits, this dual process is shown to approximate branching Brownian motion, and so the Skorokhod–McKean representation then provides a characterisation of the behaviour of the original spatial Λ-Fleming–Viot process in terms of a Fisher–KPP equation. This in turn allows for further analysis of more complicated scaling limits via parabolic PDE theory. Our main results are then Theorem 4.4, Theorem 6.2 and Theorem 6.5, describing the simple scaling limit and then two more complicated scaling limits respectively.

Given that we are trying to give a large advantage to the expanding population, then either of the more complicated scaling limits may (before knowing their limiting equation) seem physically plausible, as both give an infinitely large advantage to the expanding population over its 'competitor', empty space. In both cases, however, the limiting equation is clearly unphysical, each for different reasons. The simpler scaling limit remains a plausible option. Mathematically the investigation of all three scaling limits is of interest in its own right.

2 Definitions and Duality

We wish to model the uncontested expansion of a species into a new environment. This is represented by a function w, for which $w_t(x)$ represents the population density of the species at a point (t, x) in time and space. The population density evolves over time according to neutral and expansive events. The former represents the random fluctuations of birth and death in the species, whilst the latter drives the expansion of the species, with a parameter k specifying how fecund the species is: we expect to choose a large k to model the fact that the species may expand without opposition. The function w maps into $[0, 1]$. Thus we are implicitly assuming that there is an upper bound on the population density, and that furthermore this bound is homogeneous in space and time.

Definition 2.1 (Spatial Λ-Fleming–Viot for expansion (SLFVE))**.** Let $\mathbb{d} \in \mathbb{N}$. Let $u \in (0, 1)$, which we shall call the *impact parameter*. Let $s \in (0, 1)$, the *reproduction parameter*[1]. Let $R > 0$, the *event radius*. Let $k \in \{2, 3, 4, \ldots\}$, the *fecundity parameter*. Let $\gamma_1 > 0$, $\gamma_2 > 0$. Let Π be a Poisson point process on $[0, \infty) \times \mathbb{R}^{\mathbb{d}}$ with intensity measure $\gamma_1 \mathrm{d}t \otimes \gamma_2 \mathrm{d}x$.

Then the *SLFVE driven by* Π *with parameters* (u, s, R, k) is the process $w = (w_t)_{t \geqslant 0}$, taking values in the space of functions $\mathbb{R}^{\mathbb{d}} \to [0, 1]$, with dynamics given as follows:

For each $(t, x) \in \Pi$, a *reproduction event* occurs:

(i) With probability $1 - s$, the reproduction event is *neutral*, meaning that we pick an *offspring location* $z \in B(x, R)$ uniformly at random, let $\alpha \sim \text{Bernoulli}(w_{t-}(z))$, and then set, for each $y \in B(x, R)$,

$$w_t(y) = (1 - u)w_{t-}(y) + u\alpha.$$

[1]The reproduction parameter is equivalent to the 'selection coefficient' between two genic types in the usual population genetics literature.

(ii) With complementary probability s, the reproduction event is *expansive*, meaning that for $i \in \{1, \dots, k\}$ we pick *offspring locations* $z_i \in B(x, R)$ uniformly at random and independent of each other. Let $\alpha_i \sim \text{Bernoulli}(w_{t-}(z_i))$ independent of each other, and then set, for each $y \in B(x, R)$,

$$w_t(y) = (1 - u)w_{t-}(y) + u \max\{\alpha_i \mid i \in \{1, \dots, k\}\}.$$

We shall use the notation '$\mathbb{E}_{p_0}$', for some function $p_0 \colon \mathbb{R}^d \to [0, 1]$, to denote that $w_0 = p_0$.

The ball $B(x, R)$ is referred to as the *affected region* of a reproduction event.

Note that biologically speaking, the offspring locations actually make sense as the positions of parent individuals. Nonetheless we shall use this terminology for consistency with the dual process, for which such notions are reversed, see Remark 2.5.

Remark 2.2. Whilst we have fixed the impact parameter u and event radius R, we expect that we may in fact take these to be random, under certain conditions; we have simply chosen then constant for simplicity, see for example [EVY14] or [EFP17].

Furthermore we expect that we could take the fecundity parameter k to be random as well, subject to certain boundedness assumptions to ensure that the following analysis still holds. (That the maximum value of k is bounded would suffice.)

Having made this definition, we see that we might hope to model population expansion by taking k very large; indeed, letting k go to infinity. This would mean that every time an expansive event's affected region overlaps some region of the population then the population will expand out into the entirety of the affected region. We investigate this possibility in Section 6, and as previously remarked, discover that the results are in fact unphysical.

Moving on, we wish to find a dual process to the SLFVE. The following informal argument motivates its construction. Consider picking an individual at some point (t, x) in time and space. In some sense, with probability $w_t(x)$ the individual is alive, and with probability $1 - w_t(x)$ it is dead. In order to determine the value of $w_t(x)$, we trace back in time until the most recent reproduction event whose affected region contained x. With probability u the individual has its state of alive-or-dead determined by that reproduction event. (With complementary probability $1 - u$, we ignore this reproduction event and trace back to the next most recent reproduction event that covered the location x.) If the reproduction event was neutral then the alive-or-dead state of the individual is determined by the offspring location, whose location we pick uniformly at random from the affected region of the reproduction event. If the reproduction event is expansive then the alive-or-dead state of the individual is determined by the k offspring locations, for whom similarly we pick locations uniformly at random within the affected region of the reproduction event. For an individual at each of these offspring location(s), we repeat our above procedure, tracing backwards in time, until eventually we reach time zero. The alive-or-dead state of each of the individuals is now determined by sampling from some initial distribution p_0, which now propagates forwards in time to determine the alive-or-dead state of our initially considered individual.

Definition 2.3 (SLFVE dual)**.** Let $\mathbb{d} \in \mathbb{N}$. Let $u \in (0, 1)$, $s \in (0, 1)$, $R > 0$, $k \in \{2, 3, 4, \dots\}$, and Π be a Poisson point process on $[0, \infty) \times \mathbb{R}^{\mathbb{d}}$, as in the definition of the SLFVE.

Then the *SLFVE dual driven by* Π *with parameters* (u, s, R, k) is the $\bigcup_{n\in\mathbb{N}} (\mathbb{R}^{\mathrm{d}} \times \{0,1\})^n$-valued process $\Xi = (\Xi_t)_{t\geqslant 0}$ of individuals, each of which may be 'marked'[2], defined as follows.

We write $\Xi_t = (\xi_t^1, \dots, \xi_t^{N_t})$ for the locations of the random number $N_t \in \mathbb{N}$ of individuals at time t. We suppress in our notation whether or not the individuals are marked, and so we shall also refer to an individual by its location.

At time zero, independent of all else, each individual $\xi_0^1, \dots, \xi_0^{N_0}$ is marked with probability u. We shall usually consider the process as starting with a single individual, in which case '$\mathbb{E}_x$' denotes that the single initial individual ξ_0^1 has location $x \in \mathbb{R}^{\mathrm{d}}$.

The dynamics are as follows. For each $(t, x) \in \Pi$, a reproduction event occurs:

(i) (a) With probability $1 - s$, the reproduction event is *neutral*, meaning that if at least one individual in $B(x, R)$ is marked, then all marked individuals in $B(x, R)$ are replaced by a single *offspring individual*, whose location z is drawn uniformly at random from $B(x, R)$.

(b) With complementary probability s, the reproduction event is *expansive*, meaning that if at least one individual in $B(x, R)$ is marked, then all marked individuals in $B(x, R)$ are replaced by k *offspring individuals*, whose locations $z_1, \dots, z_k$ are drawn uniformly at random from $B(x, R)$, conditionally independent of each other.

(ii) After this, regardless of whether the reproduction event is neutral or expansive, then all individuals in $B(x, R)$ (including both any unmarked individuals who did not take part, and any new offspring individuals), are marked, independently, with probability u.

Definition 2.4. We establish some terminology referring to this dual process:

(i) That multiple marked individuals might be affected by the same reproduction event is referred to as *coalescence*.

(ii) Similar to the SLFVE, the locations $z, z_1, \dots, z_k$ are referred to as the *offspring locations*.

(iii) The marked individuals that are being replaced are referred to as the *parent individual(s)*.

(iv) We will refer to the parent individuals *dying* when they are replaced, and the offspring individuals being *born*, and to individuals being *alive* in between their birth and death times.

(v) A *lineage* is a sequence of individuals $(\zeta^0, \dots, \zeta^m)$ such that ζ^0 is the initial individual, and ζ^j is the parent of ζ^{j+1} for all $j \in \{0, \dots, m-1\}$. The lineage of a particular individual ζ is such a sequence such that $\zeta^m = \zeta$. To be a lineage *up to time* T, then ζ^m must be alive at time T.

[2] So for each value in $\bigcup_{n\in\mathbb{N}} (\mathbb{R}^{\mathrm{d}} \times \{0,1\})^n$ that the process takes, the value of n represents the number of individuals, each of which have a location in $\mathbb{R}^{\mathrm{d}}$ and are either marked ($1 \in \{0,1\}$) or unmarked ($0 \in \{0,1\}$).

(vi) We shall later show that with high probability, coalescence does not occur for our particular choice of scaling. Provided it does not occur, then neutral events will only ever 'move' individuals, by killing the parent individual and replacing them with a nearby offspring individual. In this scenario we shall instead think of these parent and offspring individuals as being the same individual, which is moving according to a pure jump process. In this case the birth and death times correspond to when the individual is produced in an expansive event, or replaced in an expansive event, respectively. Similarly we shall update our notion of lineage to being a sequence of such pure jump processes, rather than a sequence of points. Any particular individual will now also have a unique lineage.

Remark 2.5. In the usual way for such dualities, the SLFVE dual corresponds to a backwards-in-time description of the SLFVE. As a result, the notions of 'parent' and 'offspring' in the dual are the wrong way around, biologically speaking: nonetheless we use this way of talking about things for consistency with the terminology of branching Brownian motion, which the SLFVE dual will later be shown to approximate.

Definition 2.6. Some further definitions relating to the dual process:

(i) Let $\mathrm{Lin}_T(\Xi)$ denote the number of lineages in Ξ up to time T.

(ii) Let $\mathrm{Simul}_T(\Xi)$ denote the event

$$\{\text{There exist two distinct lineages in } \Xi \text{ up to time } T \text{ and } t \in [0,T] \text{ such that both lineages are marked at time } t.\}$$

with complementary event $\mathrm{NoSimul}_T(\Xi)$. If $\mathrm{NoSimul}_T(\Xi)$ holds then there is at most one marked individual at all times up to time T, which in particular means that the lineages evolve independently and that coalescence does not occur.

(iii) For $\{\zeta_i | i \in I\}$ the set of lineages in Ξ up to time T, and letting $\mathrm{Expan}_T(\zeta)$ denote the number of expansive events experienced by the lineage ζ, we may now define

$$\mathrm{Expan}_T(\Xi) = \max_{i \in I} \left[\mathrm{Expan}_T(\zeta_i)\right].$$

(iv) Let $p \colon \mathbb{R}^d \to [0,1]$. Let $t \geqslant 0$. Then the *value* of Ξ_t, with respect to p, is

$$\mathbb{V}_p(\Xi_t) = 1 - \prod_{i=1}^{N_t} (1 - p(\xi_t^i)).$$

where $\Xi_t = (\xi_t^1, \ldots, \xi_t^{N_t})$.

Remark 2.7. For $\alpha_i \sim \mathrm{Bernoulli}(p(\xi_t^i))$ for $i \in \{1, \ldots, N_t\}$, conditionally independent on Ξ_t, the value of Ξ has a probabilistic interpretation as

$$\begin{aligned}
\mathbb{V}_p(\Xi_t) &= 1 - \prod_{i=1}^{N_t} (1 - p(\xi_t^i)) \\
&= \mathbb{E}\left[1 - \prod_{i=1}^{N_t} (1 - \alpha_i) \middle| \Xi_t \right] \\
&= \mathbb{E}\left[\max\{\alpha_i \mid i \in \{1, \ldots, N_t\}\} \middle| \Xi_t \right]
\end{aligned}$$

The first equality holds by conditional independence of the α_i, and the second equality holds because each α_i takes values in $\{0, 1\}$.

Notation. For W a branching Brownian motion in $\mathbb{R}^\mathrm{d}$, we shall write $W_t = (W_t^1, \ldots, W_t^{N_t})$ for the locations of the N_t individuals present at time t. The notation '$\mathbb{E}_x$' will denote that it starts with a single individual at location x, that is, $W_0 = (W_0^1)$ with $W_0^1 = x$.

For consistency of notation, we will use the letter B for Brownian motion and the letter W for branching Brownian motion.

Proposition 2.8 (Duality between the SLFVE and the SLFVE dual)**.** *Let w and Ξ be an SLFVE and SLFVE dual, each driven by a Poisson point process with intensity measure $\gamma_1 \mathrm{d}t \otimes \gamma_2 \mathrm{d}x$, each with the same parameters (u, s, R, k). Let $p_0 \in C_c(\mathbb{R}^\mathrm{d}; [0, 1])$. Then for all $x \in \mathbb{R}^\mathrm{d}$ and $t \geqslant 0$,*

$$\mathbb{E}_{p_0}\left[w_t(x)\right] = \mathbb{E}_x\left[\mathbb{V}_{p_0}(\Xi_t)\right].$$

Remark 2.9. Note that it does not matter whether or not the SLFVE and the SLFVE dual are driven by the same or different Poisson point processes or not, as the result is about equality of expectations. In fact, from the discussion preceding the definition of the dual, we might expect that the SLFVE dual should be driven by the time-reversal of the Poisson point process driving the SLFVE. A Poisson point process has the same distribution as its time-reversal, however, so this is an unecessary complication when stating our above result.

In light of this, we might wonder if we can improve our result to get an equality without expectations, by coupling the SLFVE and the SLFVE dual in an appropriate way. This seems unlikely, however: statements of the form of equations (2.4) and (2.5), below, are likely to be the strongest that can be made. This is because the marking of individuals in the dual represents a finite sampling from an infinitely large pool; proportion u of which corresponds to the value $w_{t-}(z)$; proportion $1 - u$ corresponding to the value $w_{t-}(y)$, in Definition 2.1 of the SLFVE. In short, the SLFVE has no analogue for the marking of individuals in the SLFVE dual.

Proof of Proposition 2.8. In order to prove this result we consider a slight adaptation of the SLFVE dual process, in which there is no recording of the marking of individuals. Instead, every time a reproduction event occurs, each individual covered by the affected region of that reproduction event will independently sample from a Bernoulli(u) distribution in order to determine if they are considered marked for the purposes of the reproduction event. It is clear that this process has exactly the same dynamics as the usual SLFVE dual process, it just that whether or not an individual is marked is instead determined as events affect them, rather than before. Thus the state space is now $\bigcup_{n \in \mathbb{N}} (\mathbb{R}^\mathrm{d})^n$.

Let A_w and A_Ξ be the generators for w and Ξ respectively. Let

$$f(\widetilde{w}, \widetilde{\Xi}) = \mathbb{V}_{\widetilde{w}}(\widetilde{\Xi})$$

be defined for $\widetilde{w} \colon \mathbb{R}^\mathrm{d} \to [0, 1]$ and $\widetilde{\Xi} \in \bigcup_{n \in \mathbb{N}} (\mathbb{R}^\mathrm{d})^n$.

It is sufficient to prove that

$$(A_w f(\,\cdot\,, \widetilde{\Xi}))(\widetilde{w}) = (A_\Xi f(\widetilde{w}, \,\cdot\,))(\widetilde{\Xi}) \tag{2.1}$$

as in the usual way this implies for all fixed $t \geqslant 0$ that

$$\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{E}\left[\mathbb{V}_{w_a}(\Xi_{t-a})\middle| w_0 = p_0,\, \xi_0^1 = x\right] = 0,$$

and thus that

$$\begin{aligned}\mathbb{E}_{p_0}\left[w_t(x)\right] &= \mathbb{E}\left[\mathbb{V}_{w_t}(\Xi_0)\middle| w_0 = p_0,\, \xi_0^1 = x\right]\\ &= \mathbb{E}\left[\mathbb{V}_{w_0}(\Xi_t)\middle| w_0 = p_0,\, \xi_0^1 = x\right]\\ &= \mathbb{E}_x\left[\mathbb{V}_{p_0}(\Xi_t)\right]\end{aligned}$$

as desired.

It remains to prove equation (2.1). For $h \in (0,\infty)$, $x \in \mathbb{R}^{\mathrm{d}}$, $z \in B(x,R)$, let $C_{\mathrm{neu}}^{\widetilde{w},\widetilde{\Xi}}(h,x,z)$ be the event that

$\{w_0 = \widetilde{w},\, \Xi_0 = \widetilde{\Xi}\} \cap \{$Precisely one neutral event for the SLFVE occurs at x in the time interval $(0,h]$ with offspring location $z.\}$

And let $D_{\mathrm{neu}}^{\widetilde{w},\widetilde{\Xi}}(h,x,z)$ be defined in the same way, except that it occurs for the SLFVE dual. Similarly for $z_1,\ldots,z_k \in B(x,R)$, let $C_{\mathrm{exp}}^{\widetilde{w},\widetilde{\Xi}}(h,x,z_1,\ldots,z_k)$ be the event that

$\{w_0 = \widetilde{w},\, \Xi_0 = \widetilde{\Xi}\} \cap \{$Precisely one expansive event for the SLFVE occurs at x in the time interval $(0,h]$ with offspring locations $z_1,\ldots,z_k.\}$

with corresponding event $D_{\mathrm{exp}}^{\widetilde{w},\widetilde{\Xi}}(h,x,z_1,\ldots,z_k)$ for the SLFVE dual.

Let $\widetilde{\Xi} = (\xi^1,\ldots,\xi^N)$ and let $I_x = \{\xi^1,\ldots,\xi^N\} \cap B(x,R)$. We compute

$$\begin{aligned}&\mathbb{E}\left[\mathbb{V}_{w_h}(\widetilde{\Xi}) - \mathbb{V}_{\widetilde{w}}(\widetilde{\Xi})\middle| C_{\mathrm{neu}}^{\widetilde{w},\widetilde{\Xi}}(h,x,z)\right]\\
&= \mathbb{E}\left[\left(1 - \prod_{j\in I_x}(1-(1-u)\widetilde{w}(\xi^j) - u\alpha)\prod_{\substack{i=1\\ i\notin I_x}}^{N}(1-\widetilde{w}(\xi^i))\right) - \left(1 - \prod_{i=1}^{N}(1-\widetilde{w}(\xi^i))\right)\middle| \alpha \sim \mathrm{Bernoulli}(\widetilde{w}(z))\right]\\
&= \prod_{i=1}^{N}(1-\widetilde{w}(\xi^i)) - \widetilde{w}(z)\cdot\prod_{j\in I_x}(1-(1-u)\widetilde{w}(\xi^j) - u)\cdot\prod_{\substack{i=1\\ i\notin I_x}}^{N}(1-\widetilde{w}(\xi^i))\\
&\qquad - (1-\widetilde{w}(z))\cdot\prod_{j\in I_x}(1-(1-u)\widetilde{w}(\xi^j))\cdot\prod_{\substack{i=1\\ i\notin I_x}}^{N}(1-\widetilde{w}(\xi^i))\\
&= \prod_{\substack{i=1\\ i\notin I_x}}^{N}(1-\widetilde{w}(\xi^i))\cdot\left(\prod_{i\in I_x}(1-\widetilde{w}(\xi^i)) - \widetilde{w}(z)(1-u)^{|I_x|}\cdot\prod_{j\in I_x}(1-\widetilde{w}(\xi^j))\right.\\
&\qquad \left. - (1-\widetilde{w}(z))\cdot\prod_{j\in I_x}(1-(1-u)\widetilde{w}(\xi^j))\right)\end{aligned}$$

$$= \prod_{\substack{i=1\\ i\notin I_x}}^{N}(1-\widetilde{w}(\xi^i))\cdot\Bigg(\prod_{i\in I_x}(1-\widetilde{w}(\xi^i)) - \widetilde{w}(z)(1-u)^{|I_x|}\cdot\prod_{j\in I_x}(1-\widetilde{w}(\xi^j))$$

$$- (1-\widetilde{w}(z))\cdot\sum_{D\subseteq I_x}u^{|D|}(1-u)^{|I_x\setminus D|}\prod_{j\in I_x\setminus D}(1-\widetilde{w}(\xi^j))\Bigg)$$

$$= \prod_{\substack{i=1\\ i\notin I_x}}^{N}(1-\widetilde{w}(\xi^i))\cdot\sum_{\substack{D\subseteq I_x\\ |D|\geqslant 1}}u^{|D|}(1-u)^{|I_x\setminus D|}\Bigg(\prod_{i\in I_x}(1-\widetilde{w}(\xi^i)) - (1-\widetilde{w}(z))\cdot\prod_{j\in I_x\setminus D}(1-\widetilde{w}(\xi^j))\Bigg). \tag{2.2}$$

Furthermore,

$$\mathbb{E}\left[\mathbb{V}_{\widetilde{w}}(\Xi_h) - \mathbb{V}_{\widetilde{w}}(\widetilde{\Xi})\,\middle|\, D^{\widetilde{w},\widetilde{\Xi}}_{\mathrm{neu}}(h,x,z)\right]$$

$$= \sum_{\substack{D\subseteq I_x\\ |D|\geqslant 1}}u^{|D|}(1-u)^{|I_x\setminus D|}\left[\prod_{i=1}^{N}(1-\widetilde{w}(\xi^i)) - (1-\widetilde{w}(z))\cdot\prod_{\substack{i=1\\ i\notin D}}^{N}(1-\widetilde{w}(\xi^i))\right]$$

$$= \prod_{\substack{i=1\\ i\notin I_x}}^{N}(1-\widetilde{w}(\xi^i))\cdot\sum_{\substack{D\subseteq I_x\\ |D|\geqslant 1}}u^{|D|}(1-u)^{|I_x\setminus D|}\Bigg(\prod_{i\in I_x}(1-\widetilde{w}(\xi^i)) - (1-\widetilde{w}(z))\cdot\prod_{i\in I_x\setminus D}(1-\widetilde{w}(\xi^i))\Bigg). \tag{2.3}$$

Where the first equality holds due to our alternative interpretation of the marking of individuals in the SLFVE dual.

So we see that equations (2.2) and (2.3) are equal. That is,

$$\mathbb{E}\left[\mathbb{V}_{w_h}(\widetilde{\Xi}) - \mathbb{V}_{\widetilde{w}}(\widetilde{\Xi})\,\middle|\, C^{\widetilde{w},\widetilde{\Xi}}_{\mathrm{neu}}(h,x,z)\right] = \mathbb{E}\left[\mathbb{V}_{\widetilde{w}}(\Xi_h) - \mathbb{V}_{\widetilde{w}}(\widetilde{\Xi})\,\middle|\, D^{\widetilde{w},\widetilde{\Xi}}_{\mathrm{neu}}(h,x,z)\right]. \tag{2.4}$$

Similarly, one may check that

$$\mathbb{E}\left[\mathbb{V}_{w_h}(\widetilde{\Xi}) - \mathbb{V}_{\widetilde{w}}(\widetilde{\Xi})\,\middle|\, C^{\widetilde{w},\widetilde{\Xi}}_{\mathrm{exp}}(h,x,z_1,\ldots,z_k)\right]$$
$$= \mathbb{E}\left[\mathbb{V}_{\widetilde{w}}(\Xi_h) - \mathbb{V}_{\widetilde{w}}(\widetilde{\Xi})\,\middle|\, D^{\widetilde{w},\widetilde{\Xi}}_{\mathrm{exp}}(h,x,z_1,\ldots,z_k)\right]. \tag{2.5}$$

Now let V_R be the volume of $B(0,R)\subseteq\mathbb{R}^{\mathrm{d}}$ and let C_h be the event

$$\{\text{Precisely one reproduction event for the SLFVE occurs in the time interval } (0,h]\}.$$

with corresponding event D_h for the SLFVE dual. Then we compute

$$(A_w f(\,\cdot\,,\widetilde{\Xi}))(\widetilde{w})$$
$$= \lim_{h\to 0}\gamma_1\cdot\mathbb{E}\left[\mathbb{V}_{w_h}(\widetilde{\Xi}) - \mathbb{V}_{\widetilde{w}}(\widetilde{\Xi})\,\middle|\,\{w_0=\widetilde{w}\}\cap C_h\right]$$
$$= \gamma_1\cdot\mathbb{E}\left[\mathbb{V}_{w_h}(\widetilde{\Xi}) - \mathbb{V}_{\widetilde{w}}(\widetilde{\Xi})\,\middle|\,\{w_0=\widetilde{w}\}\cap C_h\right]$$
$$= \gamma_1\gamma_2\int_{\mathbb{R}^{\mathrm{d}}}\Bigg[\frac{1-s}{V_R}\int_{B(x,R)}\mathbb{E}\left[\mathbb{V}_{w_h}(\widetilde{\Xi}) - \mathbb{V}_{\widetilde{w}}(\widetilde{\Xi})\,\middle|\, C^{\widetilde{w},\widetilde{\Xi}}_{\mathrm{neu}}(h,x,z)\right]\,\mathrm{d}z$$
$$+ \frac{s}{(V_R)^k}\int_{(B(x,R))^k}\mathbb{E}\left[\mathbb{V}_{w_h}(\widetilde{\Xi}) - \mathbb{V}_{\widetilde{w}}(\widetilde{\Xi})\,\middle|\, C^{\widetilde{w},\widetilde{\Xi}}_{\mathrm{exp}}(h,x,z_1,\ldots,z_k)\right]\,\mathrm{d}z_1\cdots\mathrm{d}z_k\Bigg]\,\mathrm{d}x$$

$$= \gamma_1\gamma_2 \int_{\mathbb{R}^{\mathrm{d}}} \left[\frac{1-s}{V_R} \int_{B(x,R)} \mathbb{E}\left[\mathbb{V}_{\widetilde{w}}(\Xi_h) - \mathbb{V}_{\widetilde{w}}(\widetilde{\Xi}) \middle| D^{\widetilde{w},\widetilde{\Xi}}_{\mathrm{neu}}(h,x,z) \right] \mathrm{d}z \right.$$

$$\left. + \frac{s}{(V_R)^k} \int_{(B(x,R))^k} \mathbb{E}\left[\mathbb{V}_{\widetilde{w}}(\Xi_h) - \mathbb{V}_{\widetilde{w}}(\widetilde{\Xi}) \middle| D^{\widetilde{w},\widetilde{\Xi}}_{\exp}(h,x,z_1,\ldots,z_k) \right] \mathrm{d}z_1 \cdots \mathrm{d}z_k \right] \mathrm{d}x$$

$$= \gamma_1 \cdot \mathbb{E}\left[\mathbb{V}_{\widetilde{w}}(\Xi_h) - \mathbb{V}_{\widetilde{w}}(\widetilde{\Xi}) \middle| \{\Xi_0 = \widetilde{\Xi}\} \cap D_h \right]$$

$$= \lim_{h\to 0} \gamma_1 \cdot \mathbb{E}\left[\mathbb{V}_{\widetilde{w}}(\Xi_h) - \mathbb{V}_{\widetilde{w}}(\widetilde{\Xi}) \middle| \{\Xi_0 = \widetilde{\Xi}\} \cap D_h \right]$$

$$= (A_{\Xi} f(\widetilde{w}, \cdot\,))(\widetilde{\Xi}).$$

The first and final equalities follow in the usual manner from the Poisson nature of the driving Poisson point process. The second and penultimate equalities follow from the constancy of w and Ξ between reproduction events, for small enough h. The fourth equality follows from (2.4) and (2.5). Thus equation (2.1) is established. □

3 Scaling and Approximation

We consider a scaling limit in which we fix the fecundity parameter k of a sequence of SFLVEs. More complicated scaling limits, in which k can vary, for example sending $k \to \infty$, are considered in Section 6.

We begin with a sequence of technical lemmas showing that various forms of bad behaviour — large numbers of expansive events, simultaneous marking of individuals, and not approximating branching Brownian motion — occur only with small probability, which then allow us to approximate the value of the dual by an equivalent 'value' of branching Brownian motion, in Proposition 3.8. This is then used in the next section in order to deduce the first of our main theorems. Our choice of scaling limit is influenced heavily by [EFP17], from which much of this section is adapted.

Let $\mathrm{d} \in \mathbb{N}$. Let $\beta \in (0, 1/4)$. Let $u \in (0,1)$, $s \in (0,1)$, $R > 0$ and $k \in \{2,3,4,\ldots\}$. For each $n \in \mathbb{N}$, let

$$u_n = \frac{u}{n^{1-2\beta}} \qquad \text{and} \qquad s_n = \frac{s}{n^{2\beta}} \qquad \text{and} \qquad R_n = \frac{R}{n^{\beta}}.$$

Let Π^n be a Poisson point process on $[0,\infty) \times \mathbb{R}^{\mathrm{d}}$ with intensity measure $n\mathrm{d}t \otimes n^{\beta}\mathrm{d}x$. Here $n^{\beta}\mathrm{d}x$ denotes the scaling for which the *linear* scale factor of the infinitesimal region $\mathrm{d}x$ is n^{β}, and so the volume of a region is scaled by $n^{\mathrm{d}\beta}$.

For each $n \in \mathbb{N}$, let w^n be an SLFVE and Ξ^n be an SLFVE dual, in each case driven by Π^n, with parameters[3] (u_n, s_n, R_n, k).

Consider the motion of a single lineage in Ξ^n. It evolves as a pure jump process which is homogenous in both space and time. Let V_{R_n} be the volume of $B(0,R_n) \subseteq \mathbb{R}^{\mathrm{d}}$, and let $V^{(z)}_{R_n}$ be the volume of $B(0,R_n) \cap B(z,R_n)$, for $z \in \mathbb{R}^{\mathrm{d}}$. In order for the process to jump from y to $y+z$, it has to be affected by a reproduction event that covers both y and $y+z$. The volume of possible centres, x, of such reproduction events is $V^{(z)}_{R_n}$, and so the intensity with which such a centre is selected is $n \cdot n^{\mathrm{d}\beta} \cdot V^{(z)}_{R_n}$. The offspring location is

[3]Note that k does not depend on n, but every other parameter does. For the more complicated case in which k depends on n, see Section 6.

chosen uniformly at random from the ball $B(x, R_n)$, so the probability of z being chosen is $\mathrm{d}z/V_{R_n}$. Finally, the individual in our lineage must be marked to be affected, which occurs with probability u_n. Thus we see that the jump intensity m_n of the lineage is given by

$$m_n(\mathrm{d}z) = n \cdot n^{\mathbb{d}\beta} \cdot V_{R_n}^{(z)} \cdot \frac{\mathrm{d}z}{V_{R_n}} \cdot u_n = u_n n^{1+\mathbb{d}\beta} \frac{V_{R_n}^{(z)}}{V_{R_n}} \mathrm{d}z.$$

The total rate of jumps is

$$\begin{aligned}
\int_{\mathbb{R}^{\mathbb{d}}} m_n(\mathrm{d}z) &= u_n n^{1+\mathbb{d}\beta} \frac{1}{V_{R_n}} \int_{\mathbb{R}^{\mathbb{d}}} V_{R_n}^{(z)}\,\mathrm{d}z \\
&= u_n n^{1+\mathbb{d}\beta} \frac{1}{V_{R_n}} \int_{\mathbb{R}^{\mathbb{d}}} \int_{\mathbb{R}^{\mathbb{d}}} \mathbb{1}_{\{|x|<R_n\}} \mathbb{1}_{\{|x-z|<R_n\}}\,\mathrm{d}x\,\mathrm{d}z \\
&= u_n n^{1+\mathbb{d}\beta} V_{R_n} \\
&= u n^{2\beta} R^{\mathbb{d}} V_1.
\end{aligned}$$

Now $1 - s_n$ of the jumps will be from neutral events, and s_n of the jumps will be from expansive events. Thus the lineage is affected by expansive events at rate η, where

$$\eta = u n^{2\beta} R^{\mathbb{d}} V_1 s_n = u s R^{\mathbb{d}} V_1$$

which we note is independent of n. Let

$$\lambda = \eta(k-1)$$

and let

$$\sigma^2 = \frac{1}{\mathbb{d}} \int_{\mathbb{R}^{\mathbb{d}}} |z|^2 m_n(\mathrm{d}z) = \frac{u R^{\mathbb{d}} V_1}{\mathbb{d}}.$$

which is also independent of n.

Now we have some technical conditions necessary for our analysis. Let $\alpha \in (0, \min\{1-4\beta, \beta\})$, and let $b > 0$ be small enough that

$$2b \log k + 4\beta - 1 + \alpha < 0 \tag{3.1}$$

and now pick $T > 0$ small enough[4] that

$$T\eta < b, \tag{3.2}$$
$$\alpha + T\lambda < \beta, \tag{3.3}$$
$$b \log k + T\lambda + b \log\left(\frac{eT\eta}{b}\right) + \alpha < 0. \tag{3.4}$$

Finally let

$$T_n = T \log n \qquad \text{and} \qquad b_n = b \log n.$$

The first two results, Lemma 3.1 and Proposition 3.2, are adapted from Lemma 3.14 of [EFP17].

[4]Or equivalently we could impose that T u, s, R are collectively small enough that $T\lambda$ and $T\eta$ are small enough that equations (3.2)–(3.4) hold.

Lemma 3.1. *The maximal number of expansive events experienced by any given lineage is bounded in the following manner.*

$$\mathbb{P}\left[\mathrm{Expan}_{T_n}(\Xi^n) > b_n\right] = o(n^{-\alpha - T\lambda})$$

and

$$\mathbb{P}\left[\mathrm{Expan}_{T_n}(\Xi^n) > b_n \,\middle|\, \mathrm{NoSimul}_{T_n}(\Xi^n)\right] = o(n^{-\alpha - T\lambda}).$$

Proof. By equation (3.4), as $n \to \infty$,

$$(b \log k + T\lambda + b \log\left(\frac{eT\eta}{b}\right) + \alpha) \log n \to -\infty.$$

Taking an exponential yields

$$n^{b \log k} \cdot \left(\frac{eT\eta}{b}\right)^{b \log n} = o(n^{-\alpha - T\lambda}). \tag{3.5}$$

Now recall that for $Z \sim \mathrm{Poisson}(\chi)$, a Chernoff bound gives for $\alpha > \chi$ that

$$\mathbb{P}[Z > \alpha] \leqslant \frac{\mathrm{e}^{-\chi}(\mathrm{e}\chi)^\alpha}{\alpha^\alpha} \leqslant \left(\frac{\mathrm{e}\chi}{\alpha}\right)^\alpha. \tag{3.6}$$

So let $Z_n \sim \mathrm{Poisson}(T_n\eta)$. Now by equation (3.2), $b_n = b \log n > T\eta \log n = T_n\eta$, so we may apply (3.6) to deduce that

$$\mathbb{P}[Z_n > b_n] \leqslant \left(\frac{eT_n\eta}{b_n}\right)^{b_n} = \left(\frac{eT\eta}{b}\right)^{b \log n}. \tag{3.7}$$

Now let $Z = \{\zeta_i | i \in I\}$ be the set of lineages in Ξ^n up to time T_n. Define an equivalence relation $\sim$ on Z by $\zeta^{(1)} \sim \zeta^{(2)}$ if the first $1 + \min\{b_n, \mathrm{Expan}_{T_n}(\zeta^{(1)}), \mathrm{Expan}_{T_n}(\zeta^{(2)})\}$ individuals of $\zeta^{(1)}$ and $\zeta^{(2)}$ are the same. Then

$$\mathrm{Expan}_T(\Xi) = \max_{z \in Z/\sim} \max_{i \in z} \left[\mathrm{Expan}_T(\zeta_i)\right].$$

Letting $P[\,\cdot\,]$ refer to either $\mathbb{P}[\,\cdot\,]$ or $\mathbb{P}[\,\cdot \,|\, \mathrm{NoSimul}_{T_n}(\Xi^n)]$,

$$\begin{aligned}
&P\left[\mathrm{Expan}_{T_n}(\Xi^n) > b_n\right] \\
&\leqslant k^{b_n} \cdot P\left[\text{A particular lineage in } \Xi^n_{T_n} \text{ experiences more than } b_n \text{ expansive events}\right] \\
&= n^{b \log k} \cdot \mathbb{P}[Z_n > b_n] \\
&\leqslant n^{b \log k} \cdot \left(\frac{eT\eta}{b}\right)^{b \log n} \\
&= o(n^{-\alpha - T\lambda}).
\end{aligned}$$

Where the first inequality is a union bound over $Z/\sim$, the third inequality follows from equation (3.7), and the final line from equation (3.5). □

Proposition 3.2. *It is only with low probability that two individuals are simultaneously marked. Precisely,*

$$\mathbb{P}\left[\mathrm{Simul}_{T_n}(\Xi^n)\right] = o(n^{-\alpha}).$$

Proof. Consider a particular pair of lineages in Ξ^n up to time T_n. The number of reproduction events before time T_n whose affected region affects the first lineage is Poisson with mean $\Theta(nT_n) = \Theta(n \log n)$. After the reproduction event, the probability that both lineages are marked is u_n^2 (regardless of whether the second lineage was affected by the reproduction event as well). Thus the probability that both lineages are both marked at some time $t \in [0, T_n]$ is $\mathcal{O}(u_n^2 n \log n) = \mathcal{O}(n^{4\beta-1} \log n)$.

To simplify notation, for some particular choice of two lineages, let:

$$A = \{\text{There exists } t \in [0, T_n] \text{ such that two particular lineages are marked at time } t.\}$$

and let $M^n_{T_n}$ denote the number of pairs of lineages in Ξ^n up to time T_n.

Then by a union bound and Lemma 3.1,

$$\begin{aligned}
\mathbb{P}\left[\mathrm{Simul}_{T_n}(\Xi^n)\right] &\leqslant \mathbb{P}\left[\mathrm{Expan}_{T_n}(\Xi^n) > b_n\right] + \mathbb{P}\left[\mathrm{Simul}_{T_n}(\Xi^n) \,\middle|\, \mathrm{Expan}_{T_n}(\Xi^n) \leqslant b_n\right] \\
&\leqslant \mathbb{P}\left[\mathrm{Expan}_{T_n}(\Xi^n) > b_n\right] \\
&\quad + \mathbb{E}\left[M^n_{T_n} \,\middle|\, \mathrm{Expan}_{T_n}(\Xi^n) \leqslant b_n\right] \cdot \mathbb{P}\left[A \,\middle|\, \mathrm{Expan}_{T_n}(\Xi^n) \leqslant b_n\right] \\
&\leqslant o(n^{-\alpha - T\lambda}) + k^{2b_n} \cdot \frac{\mathbb{P}[A]}{\mathbb{P}\left[\mathrm{Expan}_{T_n}(\Xi^n) \leqslant b_n\right]} \\
&= o(n^{-\alpha - T\lambda}) + k^{2b_n} \cdot \frac{\mathcal{O}(n^{4\beta-1} \log n)}{1 - o(n^{-\alpha - T\lambda})} \\
&= o(n^{-\alpha}) + \mathcal{O}(k^{2b_n} \cdot n^{4\beta - 1} \log n).
\end{aligned}$$

So it suffices to show that

$$k^{2b_n} \cdot n^{4\beta-1} \cdot \log n = o(n^{-\alpha}).$$

Taking logs, this is equivalent to showing

$$(2b \log k + 4\beta - 1 + \alpha) \log n + \log \log n \to -\infty,$$

which is true by equation (3.1). □

Lemma 3.3. *The expected number of lineages in Ξ^n up to time T_n, conditioned on* $\mathrm{NoSimul}_{T_n}(\Xi^n)$*, is given by*

$$\mathbb{E}\left[\mathrm{Lin}_{T_n}(\Xi^n) \middle| \mathrm{NoSimul}_{T_n}(\Xi^n)\right] = n^{T\lambda}.$$

Proof. As we are conditioning on $\mathrm{NoSimul}_{T_n}(\Xi^n)$, then coalescence does not occur. So ignoring their spatial motion, the individuals of Ξ^n my be viewed as the vertices of a k-ary tree, with edges between parent and offspring individuals.

Similarly, ignoring the spatial motion of k-adic branching Brownian motion, we may view each branch point of k-adic branching Brownian motion as the internal vertices of a k-ary tree, with the final individuals in the branching Brownian motion as the leaves of the k-ary tree, and edges corresponding to the individual Brownian motions comprising the branching Brownian motion.

Conditioning on $\mathrm{NoSimul}_{T_n}(\Xi^n)$, then each lineage in Ξ^n experiences expansive events at rate η independently of each another. Thus the number of lineages in Ξ^n up to time T_n is the same as the number of lineages, up to time T_n, in k-adic branching Brownian motion with rate η; this is equivalently just the number of individuals at time T_n, which is given classically by $\exp(\eta(k-1)T_n) = n^{T\lambda}$. □

Corollary 3.4. *For $t \in [0, T_n]$, and letting N_t^n be the number of individuals in Ξ^n alive at time t, then*

$$\mathbb{E}\left[N_t^n \middle| \mathrm{NoSimul}_{T_n}(\Xi^n)\right] \leqslant n^{T\lambda}.$$

Proof. Provided that Ξ^n does not exhibit coalescence before time T_n, then the number of lineages in Ξ^n up to time T_n is the same as the number of individuals alive at time T_n, which is $N_{T_n}^n$. No coalescence also implies that the number of individuals in Ξ^n can never decrease, and so $N_t^n \leqslant N_{T_n}^n$. So by Lemma 3.3,

$$\begin{aligned}\mathbb{E}\left[N_t^n \middle| \mathrm{NoSimul}_{T_n}(\Xi^n)\right] &\leqslant \mathbb{E}\left[N_{T_n}^n \middle| \mathrm{NoSimul}_{T_n}(\Xi^n)\right] \\ &= \mathbb{E}\left[\mathrm{Lin}_{T_n}(\Xi^n) \middle| \mathrm{NoSimul}_{T_n}(\Xi^n)\right] \\ &= n^{T\lambda}.\end{aligned}$$

□

Lemma 3.5. *Let ξ be a jump process in $\mathbb{R}^{\mathrm{d}}$ with jump intensity $(1 - s_n)m_n$. Then there is a Brownian motion B so that for all $t \geqslant 0$ and all $x \in \mathbb{R}^{\mathrm{d}}$,*

$$\mathbb{P}_x\left[|\xi_t - B_{\sigma^2 t}| \geqslant n^{-\beta/6}\right] = \mathcal{O}(n^{-\beta}(t \vee 1)).$$

This is just Lemma 3.8 of [EFP17], with jump intensity $(1 - s_n)m_n$ instead of m_n. Note that they run their Brownian motions at rate two, hence our value of σ^2 is twice theirs.

Proposition 3.6. *There is a k-adic branching Brownian motion W^n in $\mathbb{R}^{\mathrm{d}}$ with branching rate η such that for all $t \in [0, T_n]$ and $x \in \mathbb{R}^{\mathrm{d}}$,*

$$\mathbb{P}_x\left[\bigcup_{i=1}^{N_t^n}\left\{\left|\xi_t^{n,i} - W_{\sigma^2 t}^{n,i}\right| \geqslant 4n^{-\beta/6}b_n\right\} \middle| \mathrm{NoSimul}_{T_n}(\Xi^n)\right] = o(n^{-\alpha})$$

where $\Xi_t^n = (\xi_t^{n,1}, \ldots, \xi_t^{n,N_t^n})$ and $W_{\sigma^2 t}^n = (W_{\sigma^2 t}^{n,1}, \ldots, W_{\sigma^2 t}^{n,N_t^n})$.

(In particular Ξ_t^n and $W_{\sigma^2 t}^n$ always have the same number N_t^n of individuals.)

Proof. Ignoring the underlying Poisson point process event-based model that is driving Ξ^n, and noting that we are conditioning on there never being two or more simultaneously marked individuals, which in particular implies that lineages evolve independently, then we shall simply think of Ξ^n as a branching jump process performing jumps with intensity $(1 - s_n)m_n$ and branching with rate η. We condition on all of the branching events (at what time they occurred, and which individual was affected) up to T_n. (We are not conditioning on any individual branching event, but rather on some entire collection of branching events).

So let A be any such event corresponding to such a collection of branching events. Until otherwise specified later in this proof, we will be conditioning on this event having occurred. Consider the behaviour of Ξ^n. The initial individual will jump around until it reaches the end of its lifespan, as specified by A, and then branch into several offspring individuals: each of these will then jump around until they reach the end of their lifespans, again as specified by A, and they in turn branch: this goes on until eventually the final time T_n is reached.

Consider the lifetime of a particular individual; call this individual ζ. It will be born at some time ι and die[5] at some later time τ. Note that ι and τ are both completely specified by A. In the mean time it evolves according to a pure jump process with jump intensity $(1-s_n)m_n$. Normalise the process to start from the origin at time zero: that is, consider the process $t \mapsto \zeta_{t+\iota} - \zeta_\iota$. Then we know from Lemma 3.5 that there is a Brownian motion B^ζ which approximates this normalised pure jump process in the manner described there.

Now we may simply define our branching Brownian motion as being the appropriate composition of these Brownian motions: let χ^0 be the initial individual in Ξ^n, with death time τ_0. Then for $t \in [0, \tau_0)$, define $W^n_{\sigma^2 t}$ conditional on the event A as[6] being the location of the single individual at $x + B^{\chi^0}_{\sigma^2 t}$, where x is the initial location of χ^0. Eventually χ^0 will branch, say into the individuals $\chi^1, \ldots, \chi^k$. Then until each of those individuals branches, define $W^n_{\sigma^2 t}$ as being the locations of the individuals at $x + B^{\chi^0}_{\sigma^2 \tau_0} + B^{\chi^1}_{\sigma^2(t-\tau_0)}, \ldots, x + B^{\chi^0}_{\sigma^2\tau_0} + B^{\chi^k}_{\sigma^2(t-\tau_0)}$.

Continue to repeat in the obvious fashion. Thus, for a particular lineage $(\zeta^0, \ldots, \zeta^m)$ up to time t, with birth times $\iota_0, \ldots, \iota_m$ and death times $\tau_0, \ldots, \tau_m$, respectively[7], then the individual in the corresponding lineage of W^n at time $\sigma^2 t$ will have location $W'_{\sigma^2 t}$ given by

$$W'_{\sigma^2 t} = x + \left(\sum_{j=0}^{m-1} B^{\zeta^j}_{\sigma^2(\tau_j - \tau_{j-1})} \right) + B^{\zeta^m}_{\sigma^2(t-\tau_{m-1})} \tag{3.8}$$

where we set $\tau_{-1} = 0$.

We are now no longer conditioning on A. We have now defined our branching Brownian motion. Note that so far, technically speaking, we have only constructed a Brownian motion which undergoes branching, rather than the canonical Markovian branching Brownian motion in which the wait time before branching is exponential. But that the wait times behave in this manner is now immediate from our construction, due to our conditioning on collections of branching events[8]: as Ξ^n is branching with exponential wait times, and as W^n shares its branch times, then W^n also has exponential wait times, and is therefore such a canonical branching Brownian motion.

Finally we show that this branching Brownian motion approximates Ξ^n in the manner that we wish. Consider a particular lineage $(\xi^0, \ldots, \xi^m)$ up to time t, with birth times $\iota_0, \ldots, \iota_m$ and death times $\tau_0, \ldots, \tau_m$, respectively. First note that for all $j \in \{1, \ldots, m\}$ that $\left|\xi^{j-1}_{\tau_{j-1}} - \xi^j_{\iota_j}\right| \leqslant R_n$, and thus for all $q \in [\iota_j, \tau_j]$ that

$$\begin{aligned}
&\mathbb{P}_x \left[\left| \xi^j_q - \xi^{j-1}_{\tau_{j-1}} - B^{\zeta^j}_{\sigma^2(q-\tau_{j-1})} \right| \geqslant 2n^{-\beta/6} \middle| \operatorname{NoSimul}_{T_n}(\Xi^n) \right] \\
&\leqslant \mathbb{P}_x \left[\left| \xi^j_q - \xi^j_{\iota_j} - B^{\zeta^j}_{\sigma^2(q-\tau_{j-1})} \right| \geqslant n^{-\beta/6} \middle| \operatorname{NoSimul}_{T_n}(\Xi^n) \right] \\
&= \mathcal{O}(n^{-\beta}(q \vee 1)) \\
&= \mathcal{O}(n^{-\beta} \log n).
\end{aligned} \tag{3.9}$$

The inequality holds for all n large enough that $R_n \leqslant n^{-\beta/6}$, recalling that $R_n = \Theta(n^{-\beta})$.

[5]Or equivalently we hit our final time T_n.

[6]Events of this type partition the region of probability space in which there is no coalescence, so W^n is well defined. We define W^n arbitrarily on the remainder of the probability space.

[7]Noting that $\iota_j = \tau_{j-1}$ for each j.

[8]That is, events of the same type as A.

The first equality holds by Lemma 3.5, and the final equality holds because $q \leqslant \tau_j \leqslant T_n = \Theta(\log n)$.

This then implies by Lemma 3.1 that

$$\begin{aligned}
&\mathbb{P}_x\left[\left|\xi_q^j - \xi_{\tau_{j-1}}^{j-1} - B_{\sigma^2(q-\tau_{j-1})}^{\zeta_j}\right| \geqslant 2n^{-\beta/6} \middle| \operatorname{NoSimul}_{T_n}(\Xi^n) \cap \{\operatorname{Expan}_{T_n}(\Xi^n) \leqslant b_n\}\right] \\
&\leqslant \frac{\mathbb{P}_x\left[\left|\xi_q^j - \xi_{\tau_{j-1}}^{j-1} - B_{\sigma^2(q-\tau_{j-1})}^{\zeta_j}\right| \geqslant 2n^{-\beta/6} \middle| \operatorname{NoSimul}_{T_n}(\Xi^n)\right]}{\mathbb{P}\left[\operatorname{Expan}_{T_n}(\Xi^n) \leqslant b_n \middle| \operatorname{NoSimul}_{T_n}(\Xi^n)\right]} \\
&= \frac{\mathcal{O}(n^{-\beta}\log n)}{1 - o(n^{-\alpha - T\lambda})} \\
&= \mathcal{O}(n^{-\beta}\log n).
\end{aligned} \tag{3.10}$$

Which in turn implies that

$$\begin{aligned}
&\mathbb{P}_x\left[\sum_{j=0}^{m-1}\left|\xi_{\tau_j}^j - \xi_{\tau_{j-1}}^j - B_{\sigma^2(\tau_j-\tau_{j-1})}^{\zeta_j}\right| \geqslant 2n^{-\beta/6}b_n \middle| \operatorname{NoSimul}_{T_n}(\Xi^n)\right] \\
&\leqslant \mathbb{P}_x\left[\bigcup_{j=0}^{m-1}\left\{\left|\xi_{\tau_j}^j - \xi_{\tau_{j-1}}^j - B_{\sigma^2(\tau_j-\tau_{j-1})}^{\zeta_j}\right| \geqslant 2n^{-\beta/6}b_n/m\right\} \middle| \operatorname{NoSimul}_{T_n}(\Xi^n)\right] \\
&\leqslant \mathbb{P}_x\left[\bigcup_{j=0}^{m-1}\left\{\left|\xi_{\tau_j}^j - \xi_{\tau_{j-1}}^j - B_{\sigma^2(\tau_j-\tau_{j-1})}^{\zeta_j}\right| \geqslant 2n^{-\beta/6}b_n/m\right\} \middle| \operatorname{NoSimul}_{T_n}(\Xi^n) \cap \{\operatorname{Expan}_{T_n}(\Xi^n) \leqslant b_n\}\right] \\
&\quad + \mathbb{P}\left[\operatorname{Expan}_{T_n}(\Xi^n) > b_n \middle| \operatorname{NoSimul}_{T_n}(\Xi^n)\right] \\
&\leqslant \mathbb{E}_x\left[m \middle| \operatorname{NoSimul}_{T_n}(\Xi^n) \cap \{\operatorname{Expan}_{T_n}(\Xi^n) \leqslant b_n\}\right] \\
&\quad \cdot \mathbb{P}_x\left[\left|\xi_{\tau_j}^j - \xi_{\tau_{j-1}}^j - B_{\sigma^2(\tau_j-\tau_{j-1})}^{\zeta_j}\right| \geqslant 2n^{-\beta/6}b_n/m \middle| \operatorname{NoSimul}_{T_n}(\Xi^n) \cap \{\operatorname{Expan}_{T_n}(\Xi^n) \leqslant b_n\}\right] \\
&\quad + \mathbb{P}\left[\operatorname{Expan}_{T_n}(\Xi^n) > b_n \middle| \operatorname{NoSimul}_{T_n}(\Xi^n)\right] \\
&\leqslant b_n \cdot \mathcal{O}(n^{-\beta}\log n) + o(n^{-\alpha-T\lambda}) \\
&= o(n^{-\alpha-T\lambda}).
\end{aligned} \tag{3.11}$$

The final inequality holds by noting that $m \leqslant \operatorname{Expan}_{T_n}(\Xi^n)$ and applying equation (3.10) for the first term, and by Lemma 3.1 for the second term. The final equality holds by equation (3.3).

Now let $\tau_{-1} = 0$ and $W'_{\sigma^2 t}$ be as in equation (3.8). Then by equations (3.9) and (3.11),

$$\begin{aligned}
&\mathbb{P}_x\left[|\xi_t^m - W'_{\sigma^2 t}| \geqslant 2n^{-\beta/6}(b_n+1) \middle| \operatorname{NoSimul}_{T_n}(\Xi^n)\right] \\
&= \mathbb{P}_x\Bigg[\Bigg|\left(\xi_t^m - \xi_{\tau_{m-1}}^{m-1} + \left(\sum_{j=0}^{m-1}\xi_{\tau_j}^j - \xi_{\tau_{j-1}}^j\right) + x\right) \\
&\qquad - \left(x + \left(\sum_{j=0}^{m-1}B_{\sigma^2(\tau_j-\tau_{j-1})}^{\zeta_j}\right) + B_{\sigma^2(t-\tau_{m-1})}^{\zeta_m}\right)\Bigg| \geqslant 2n^{-\beta/6}(b_n+1) \,\Bigg|\, \operatorname{NoSimul}_{T_n}(\Xi^n)\Bigg]
\end{aligned}$$

$$\begin{aligned}
&\leqslant \mathbb{P}_x\left[\left|\xi^m_t - \xi^{m-1}_{\tau_{m-1}} - B^{\zeta_m}_{\sigma^2(t-\tau_{m-1})}\right| \geqslant 2n^{-\beta/6} \,\middle|\, \mathrm{NoSimul}_{T_n}(\Xi^n)\right]\\
&\quad + \mathbb{P}_x\left[\sum_{j=0}^{m-1}\left|\xi^j_{\tau_j} - \xi^j_{\tau_{j-1}} - B^{\zeta_j}_{\sigma^2(\tau_j-\tau_{j-1})}\right| \geqslant 2n^{-\beta/6}b_n \,\middle|\, \mathrm{NoSimul}_{T_n}(\Xi^n)\right]\\
&= \mathcal{O}(n^{-\beta}\log n) + o(n^{-\alpha-T\lambda})\\
&= o(n^{-\alpha-T\lambda})
\end{aligned}$$

So by a union bound,

$$\begin{aligned}
&\mathbb{P}_x\left[\bigcup_{i=1}^{N^n_t}\left\{\left|\xi^{n,i}_t - W^{n,i}_{\sigma^2 t}\right| \geqslant 4n^{-\beta/6}b_n\right\} \,\middle|\, \mathrm{NoSimul}_{T_n}(\Xi^n)\right]\\
&\leqslant \mathbb{P}_x\left[\bigcup_{i=1}^{N^n_t}\left\{\left|\xi^{n,i}_t - W^{n,i}_{\sigma^2 t}\right| \geqslant 2n^{-\beta/6}(b_n+1)\right\} \,\middle|\, \mathrm{NoSimul}_{T_n}(\Xi^n)\right]\\
&\leqslant \mathbb{E}_x\left[N^n_t \middle|\ \mathrm{NoSimul}_{T_n}(\Xi^n)\right]\cdot o(n^{-\alpha-T\lambda})\\
&\leqslant n^{T\lambda}\cdot o(n^{-\alpha-T\lambda})\\
&= o(n^{-\alpha})
\end{aligned}$$

with the first inequality following for n large enough that $b_n \geqslant 1$, and the final inequality following from Corollary 3.4. □

Lemma 3.7. *Let $M \in \mathbb{N}$. Let $p_0 \in C_c(\mathbb{R}^d; [0,1])$ have modulus of continuity ω. Note that p_0 is uniformly continuous so $\omega(\delta) \to 0$ as $\delta \to 0$. Then for all δ small enough that $\omega(\delta) \leqslant 1$, the function $\delta \mapsto M\omega(\delta)$ is a modulus of continuity for*

$$(z_1,\ldots,z_M) \mapsto \prod_{i=1}^M (1 - p_0(z_i))$$

with respect to the ∞-norm on $(\mathbb{R}^d)^M$.

Proof. Let $\delta > 0$ be small enough that $\omega(\delta) \leqslant 1$. Let $(z^{(1)}_1,\ldots,z^{(1)}_M), (z^{(2)}_1,\ldots,z^{(2)}_M) \in (\mathbb{R}^d)^M$ be such that $\left\|(z^{(1)}_1,\ldots,z^{(1)}_M) - (z^{(2)}_1,\ldots,z^{(2)}_M)\right\|_\infty \leqslant \delta$, meaning that

$$\left|z^{(1)}_i - z^{(2)}_i\right| \leqslant \delta \tag{3.12}$$

for each $i \in \{1,\ldots,M\}$. (Where $|\cdot|$ denotes the usual Euclidean norm.)

Now for each $i \in \{1,\ldots,M\}$, let $w^{(1)}_i = 1 - p_0(z^{(1)}_i)$ and $w^{(2)}_i = 1 - p_0(z^{(2)}_i)$, so that

$$\left|w^{(1)}_i - w^{(2)}_i\right| = \left|p_0(z^{(2)}_i) - p_0(z^{(1)}_i)\right| \leqslant \omega(\delta) \tag{3.13}$$

by equation (3.12).

Hence

$$\begin{aligned}
\prod_{i=1}^M (1 - p_0(z^{(1)}_i)) - \prod_{i=1}^M (1 - p_0(z^{(2)}_i)) &= \prod_{i=1}^M w^{(1)}_i - \prod_{i=1}^M w^{(2)}_i\\
&\leqslant \prod_{i=1}^M (w^{(2)}_i + \omega(\delta)) - \prod_{i=1}^M w^{(2)}_i\\
&\leqslant M\omega(\delta)
\end{aligned}$$

where the first inequality follows from equation (3.13), and the second inequality follows because $\omega(\delta) \leqslant 1$ and $w_i^{(2)} \leqslant 1$.

Reversing the roles of $z_i^{(1)}$ and $z_i^{(2)}$ completes the proof. □

Proposition 3.8. *Let $p_0 \in C_c(\mathbb{R}^d; [0,1])$ have modulus of continuity ω. Let W^n be as in Proposition 3.6. Then for $t \in [0, T_n]$ and $x \in \mathbb{R}^d$,*

$$\left|\mathbb{E}_x\left[\prod_{i=1}^{N_t^n}(1-p_0(\xi_t^{n,i}))\right] - \mathbb{E}_x\left[\prod_{i=1}^{N_t^n}(1-p_0(W_{\sigma^2 t}^{n,i}))\right]\right| \\ = \mathcal{O}\left(n^{-\alpha} + n^{T\lambda+\alpha}\cdot\omega\left(4n^{-\beta/6}b_n\right)\right).$$

Proof. To simplify notation, let

$$D_1 = \bigcap_{i=1}^{N_t^n}\left\{\left|\xi_t^{n,i} - W_{\sigma^2 t}^{n,i}\right| \leqslant 4n^{-\beta/6}b_n\right\}$$
$$D_2 = \left\{N_t^n \leqslant n^{T\lambda+\alpha}\right\}.$$

Now by Markov's inequality, Proposition 3.6 and Corollary 3.4,

$$\begin{aligned}\mathbb{P}\left[D_2^c \mid \mathrm{NoSimul}_{T_n}(\Xi^n)\cap D_1\right] &\leqslant \frac{\mathbb{P}\left[D_2^c \mid \mathrm{NoSimul}_{T_n}(\Xi^n)\right]}{\mathbb{P}\left[D_1 \mid \mathrm{NoSimul}_{T_n}(\Xi^n)\right]} \\ &\leqslant \frac{\mathbb{E}\left[N_t^n \mid \mathrm{NoSimul}_{T_n}(\Xi^n)\right]\cdot n^{-(T\lambda+\alpha)}}{1-o(n^{-\alpha})} \\ &\leqslant 2n^{-\alpha}. \end{aligned} \tag{3.14}$$

Where the last inequality holds for n large enough that the denominator is smaller than 1/2.

Now it is elementary that for any random variable X such that $0 \leqslant X \leqslant 1$, any event Y, any $\varepsilon \in (0,1)$, and any event Z such that $\mathbb{P}[Z \mid Y] \geqslant 1-\varepsilon$, that

$$|\mathbb{E}[X \mid Y] - \mathbb{E}[X \mid Y \cap Z]| \leqslant \varepsilon. \tag{3.15}$$

So let $X_1 = \prod_{i=1}^{N_t}(1-p_0(\xi_t^{n,i}))$ and $X_2 = \prod_{i=1}^{N_t}(1-p_0(W_{\sigma^2 t}^{n,i}))$. Then for $j \in \{1,2\}$, by Proposition 3.2 and equation (3.15),

$$|\mathbb{E}_x[X_j] - \mathbb{E}_x[X_j \mid \mathrm{NoSimul}_{T_n}(\Xi^n)]| = o(n^{-\alpha}).$$

And for $j \in \{1,2\}$, by Proposition 3.6 and equation (3.15),

$$|\mathbb{E}_x[X_j \mid \mathrm{NoSimul}_{T_n}(\Xi^n)] - \mathbb{E}_x[X_j \mid \mathrm{NoSimul}_{T_n}(\Xi^n)\cap D_1]| = o(n^{-\alpha}).$$

And for $j \in \{1,2\}$, by equation (3.14) and equation (3.15),

$$|\mathbb{E}_x[X_j \mid \mathrm{NoSimul}_{T_n}(\Xi^n)\cap D_1] - \mathbb{E}_x[X_j \mid \mathrm{NoSimul}_{T_n}(\Xi^n)\cap D_1\cap D_2]| \leqslant 2n^{-\alpha}.$$

Next, by Lemma 3.7, the fact that we are conditioning on D_1 and D_2, and for n large enough that $\omega(4n^{-\beta/6}b_n) \leqslant 1$,

$$\begin{aligned}&|\mathbb{E}_x[X_1 \mid \mathrm{NoSimul}_{T_n}(\Xi^n)\cap D_1\cap D_2] - \mathbb{E}_x[X_2 \mid \mathrm{NoSimul}_{T_n}(\Xi^n)\cap D_1\cap D_2]| \\ &\leqslant \mathbb{E}_x[|X_1 - X_2| \mid \mathrm{NoSimul}_{T_n}(\Xi^n)\cap D_1\cap D_2] \\ &\leqslant n^{T\lambda+\alpha}\cdot\omega(4n^{-\beta/6}b_n)\end{aligned}$$

And so bringing these inequalities together gives the result. □

4 Convergence to the Fisher–KPP equation

With Proposition 3.8 in hand, we are now ready to announce the result of the scaling limit, which is done in Theorem 4.4: we find that the expectation of the SLFVE converges uniformly to the solution of a Fisher–KPP equation.

We begin by stating some classical theory, describing the relationship between branching Brownian motion and the Fisher–KPP equation via the Skorokhod–McKean representation.

Definition 4.1. We will need the following anisotropic space, with differing smoothness in t and x. Although there is no standard notation for such a space, we shall use the following notation, in line with [Eva10]. Let $A \subseteq \mathbb{R}$. Then we let

$$C^2_1([0,\infty) \times \mathbb{R}^{\mathrm{d}}; A) = \{u \colon [0,\infty) \times \mathbb{R}^{\mathrm{d}} \to A \,|\, u \text{ is continuous in } [0,\infty) \times \mathbb{R}^{\mathrm{d}}, \text{ and } \nabla u, \nabla^2 u, \partial_t u \text{ exist and are continuous in } (0,\infty) \times \mathbb{R}^{\mathrm{d}}\}$$

where ∇ and ∇^2 involve spatial derivatives only, and ∇^2 denotes the Hessian.

The next theorem is classical.

Theorem 4.2. *Let W be a k-adic branching Brownian motion in $\mathbb{R}^{\mathrm{d}}$ with branching rate η. Let $g \in C(\mathbb{R}^{\mathrm{d}}; [0,1])$. Let $u \colon [0,\infty) \times \mathbb{R}^{\mathrm{d}} \to \mathbb{R}$ be defined by*

$$u(t,x) = \mathbb{E}_x \left[\prod_{i=1}^{N_t} g(W^i_t) \right]$$

where $W_t = (W^1_t, \ldots, W^{N_t}_t)$.

Then $u \in C^2_1([0,\infty) \times \mathbb{R}^{\mathrm{d}}; [0,1])$ and is the unique solution (in this space) to the Fisher–KPP equation:

$$\begin{cases} \partial_t u = \frac{1}{2}\Delta u + \eta(u^k - u), \\ u(0,x) = g(x). \end{cases}$$

Existence and uniqueness of a solution in $C^2_1([0,\infty) \times \mathbb{R}^{\mathrm{d}}; [0,1])$ is stated in [AW78, Section 2]. The probabilistic representation of solutions is known as the Skorokhod–McKean representation, see [Sko64] and [McK75].

Corollary 4.3. *Let W be a k-adic branching Brownian motion in $\mathbb{R}^{\mathrm{d}}$ with branching rate η. Let $p_0 \in C(\mathbb{R}^{\mathrm{d}}; [0,1])$. Let $v \colon [0,\infty) \times \mathbb{R}^{\mathrm{d}} \to \mathbb{R}$ be defined by*

$$v(t,x) = 1 - \mathbb{E}_x \left[\prod_{i=1}^{N_t} \big(1 - p_0(W^i_t)\big) \right]$$

where $W_t = (W^1_t, \ldots, W^{N_t}_t)$.

Then $v \in C^2_1([0,\infty) \times \mathbb{R}^{\mathrm{d}}; [0,1])$ and is the unique solution (in this space) to the Fisher–KPP equation

$$\begin{cases} \partial_t v = \frac{1}{2}\Delta v + \eta(1-v)(1-(1-v)^{k-1}), \\ v(0,x) = p_0(x) \end{cases}$$

Proof. Let $u = 1 - v$ and $g = 1 - p_0$. Then u and g are as in Theorem 4.2 and the result follows. $\square$

We may now assemble our results to deduce:

Theorem 4.4. *Let $\mathbb{d} \in \mathbb{N}$. Let $\beta \in (0, 1/4)$. Let $\alpha \in (0, \min\{1 - 4\beta, \beta\})$. Let $u \in (0, 1)$, $s \in (0, 1)$, $R > 0$ and $k \in \{2, 3, 4, \ldots\}$. Let V_1 be the volume of the ball $B(0, R) \subseteq \mathbb{R}^{\mathbb{d}}$. Let $\eta = usR^{\mathbb{d}}V_1$.*

For each $n \in \mathbb{N}$, let

$$u_n = \frac{u}{n^{1-2\beta}} \qquad \text{and} \qquad s_n = \frac{s}{n^{2\beta}} \qquad \text{and} \qquad R_n = \frac{R}{n^{\beta}}.$$

Also let

$$\sigma^2 = \frac{uR^{\mathbb{d}}V_1}{\mathbb{d}}.$$

Let Π^n be a Poisson point process on $[0, \infty) \times \mathbb{R}^{\mathbb{d}}$ with intensity measure $n\mathrm{d}t \otimes n^{\beta}\mathrm{d}x$. Let $p_0 \in C_c(\mathbb{R}^{\mathbb{d}}; [0, 1])$ have modulus of continuity ω.

For each $n \in \mathbb{N}$, let w^n be an SLFVE driven by Π^n with parameters (u_n, s_n, R_n, k). Let $v \in C_1^2([0, \infty) \times \mathbb{R}^{\mathbb{d}}; [0, 1])$ satisfy the Fisher–KPP equation

$$\begin{cases} \partial_t v = \frac{1}{2}\Delta v + \eta(1 - v)(1 - (1 - v)^{k-1}), \\ v(0, x) = p_0(x). \end{cases}$$

Then there exists $T > 0$, $b > 0$ both sufficiently small so that for all $t \in [0, T \log n]$ and $x \in \mathbb{R}^{\mathbb{d}}$,

$$\left|\mathbb{E}_{p_0}\left[w_t^n(x)\right] - v(\sigma^2 t, x)\right| = \mathcal{O}\left(n^{-\alpha} + n^{T\lambda + \alpha} \cdot \omega\left(4n^{-\beta/6} b \log n\right)\right).$$

Proof. Corollary 4.3 gives that

$$v(\sigma^2 t, x) = 1 - \mathbb{E}_x\left[\prod_{i=1}^{N_t^n} \left(1 - p_0(W_{\sigma^2 t}^{n,i})\right)\right].$$

Combining this with Proposition 2.8, the definition of $\mathbb{V}_{p_0}(\Xi_t^n)$, and Proposition 3.8, gives the result. $\square$

We see that precisely how good our approximation is depends on the regularity of p_0, in particular on the behaviour of its modulus of continuity. Reasonably mild conditions on p_0 are sufficient to give good results:

Corollary 4.5. *Assume as in Theorem 4.4. Assume further that p_0 is γ-Hölder continuous, and that $\alpha < \gamma\beta/12$. Then there exists $T > 0$ sufficiently small so that for all $t \in [0, T \log n]$ and $x \in \mathbb{R}^{\mathbb{d}}$,*

$$\left|\mathbb{E}_{p_0}\left[(w_t^n(x)\right] - v(\sigma^2 t, x)\right| = \mathcal{O}\left(n^{-\alpha}\right).$$

Proof. If p_0 is γ-Hölder continuous then $\omega(\delta) = \mathcal{O}(\delta^{\gamma})$. Apply Theorem 4.4, and impose the further smallness condition on T that $2\alpha + T\lambda < \gamma\beta/6$, which is possible by the extra restriction on α. Then

$$\begin{aligned} n^{T\lambda + \alpha} \cdot \omega\left(4n^{-\beta/6} b \log n\right) &= \mathcal{O}\left(n^{T\lambda + \alpha - \gamma\beta/6} \cdot (\log n)^{\gamma}\right) \\ &= o(n^{-\alpha}). \end{aligned}$$

$\square$

Remark 4.6. We can nearly deduce an even better result: it is an easy adaptation of Proposition 3.8 to get that the same bound holds for the expectation of the absolute value of the difference, which can then be applied in the context of Corollary 4.5, say, to deduce that

$$\mathbb{E}_x\left[\left|\mathbb{V}_{p_0}(\Xi^n_t) - v(\sigma^2 t, x)\right|\right] = \mathcal{O}(n^{-\alpha}).$$

This may now be used to give bounds on how far the SLFVE dual can deviate from its expected value, via Markov's inequality. The duality between w^n and Ξ^n is not, however, strong enough to then extend this same statement to w^n.

From Theorem 4.4 and Corollary 4.5 we may now deduce further results about the behaviour of the SLFVE. For example, from [AW78],

Theorem 4.7. *Let $F \in C^1([0,1])$ be such that $F(0) = F(1) = 0$, that $F(u) > 0$ for $u \in (0,1)$, and satisfy the KPP assumption*

$$F(u) \leqslant F'(0)u$$

for $u \in [0,1]$.

Let $h \in C_c(\mathbb{R}^{\mathrm{d}}; [0,1])$ be nontrivial. Let $v \in C^2_1([0,\infty) \times \mathbb{R}^{\mathrm{d}}; [0,1])$ satisfy the Fisher–KPP equation:

$$\begin{cases} \partial_t v = \frac{1}{2}\Delta v + F(v), \\ v(0,x) = h(x). \end{cases}$$

Then v represents the propagation, at asymptotic speed $c^* = \sqrt{2F'(0)}$, of a transition zone between the cases $v = 0$ and $v = 1$. Put precisely, this means that it enjoys the following asymptotic speed property: that for all $y \in \mathbb{R}^{\mathrm{d}}$ and all $c > c^*$,

$$\lim_{t\to\infty} \sup_{|x-y|\geqslant ct} v(t,x) = 0,$$

and for all $y \in \mathbb{R}^{\mathrm{d}}$ and all $c \in [0, c^*)$,

$$\lim_{t\to\infty} \inf_{|x-y|\leqslant ct} v(t,x) = 1.$$

The first statement follows from [AW78, Theorem 5.1], the second statement follows from[AW78, Corollary 1 to Theorem 5.3], and the value of c^* follows from [AW78, Proposition 4.2] and the KPP assumption. Note that the value of our critical speed c^* is slightly different to [AW78], as we have a coefficient of $1/2$ in front of the Laplacian; we have a scaling of space.

And thus we may deduce:

Corollary 4.8. *Assume as in Corollary 4.5. Let $c^* = \sigma^2\sqrt{2\lambda}$. Then for all $y \in \mathbb{R}^{\mathrm{d}}$, $c > c^*$ and $c' \in [0, c^*)$,*

$$\lim_{\substack{n\to\infty \\ t\to\infty}} \sup_{|x-y|\geqslant ct} \mathbb{E}_{p_0}\left[w^n_t(x)\right] = 0,$$
$$\lim_{\substack{n\to\infty \\ t\to\infty}} \inf_{|x-y|\leqslant c't} \mathbb{E}_{p_0}\left[w^n_t(x)\right] = 1,$$

where the limits must be taken together: precisely, there exist $\tau > 0$ and $N > \mathrm{e}^{\tau/T}$ that the approximation holds for all $n \geqslant N$ and all $t \in [\tau, T\log n]$.[9]

[9] This is topologically a triangle in (n,t)-space.

Proof. Fix $\varepsilon > 0$. Let $F(u) = \eta(1-u)(1-(1-u)^{k-1})$. Then $F'(0) = \eta(k-1) = \lambda$ and it is an easy computation to check that F satisfies the KPP assumption. So apply Theorem 4.7 to deduce that for any $y \in \mathbb{R}^{\mathrm{d}}$, $c/\sigma^2 > \sqrt{2\lambda}$ and $c'/\sigma^2 \in [0, \sqrt{2\lambda})$, there exists $\tau > 0$ such that for all $t \geqslant \tau$,

$$\sup_{|x-y| \geqslant \frac{c}{\sigma^2}\cdot\sigma^2 t} v(\sigma^2 t, x) > \frac{\varepsilon}{2},$$

$$\inf_{|x-y| \leqslant \frac{c'}{\sigma^2}\cdot\sigma^2 t} v(\sigma^2 t, x) < 1 - \frac{\varepsilon}{2}.$$

Now pick $N \in \mathbb{N}$ large enough that $T \log N \geqslant \tau$ and that the right hand side of Corollary 4.5 is less than $\varepsilon/2$ to deduce that for all $n \geqslant N$ and all $t \in [\tau, T \log n]$,

$$\sup_{|x-y| \geqslant \frac{c}{\sigma^2}\cdot\sigma^2 t} \mathbb{E}_{p_0}\left[w_t^n(x)\right] > \varepsilon,$$

$$\inf_{|x-y| \leqslant \frac{c'}{\sigma^2}\cdot\sigma^2 t} \mathbb{E}_{p_0}\left[w_t^n(x)\right] < 1 - \varepsilon.$$

□

Having seen that the speed of propagation is $\sqrt{2\lambda} = \sqrt{2\eta(k-1)}$, we might expect that a more complicated scaling limit worthy of investigation would be to send $\eta \to 0$ and $k \to \infty$ such that $\eta(k-1) = \mathcal{O}(1)$. This will be contained in the 'small η' case of the next two sections, in which we see that in this case we end up with convergence to the heat equation. One point of potential confusion that is worth resolving: the heat equation is often said to have infinite speed of propagation, whilst we have just stated we will be investigating a limit of equations which all have the same constant speed of propagation. How is this consistent? The answer is that the heat equation has infinite speed of propagation *of information*, which in fact the Fisher–KPP equation exhibits as well: from compact initial data, the solution nonetheless becomes nonzero everywhere, instantly. The speed property of Theorem 4.7 is that of a wave speed.

5 Convergence of solutions to Fisher–KPP equations

In some sense, the previous section completes our investigation, as the scaling limit is shown to converge to the solution of a differential equation. But what about more complicated scaling limits? In the scaling limit so far investigated, there remain two free parameters, namely η and k. Now we investigate what would happen if we wanted to send $\eta \to 0$ or $k \to \infty$. This could be after our previous scaling limit, or alongside it, by having η and k depending on n. In this section we cover the former approach by means of classical parabolic theory.

This section is essentially a technical one; the more biologically motivated reader may wish to skip to Section 6 in which we apply these results to analyse more complicated scaling limits.

We begin by stating a very general version of the Tychonoff Uniquenesss Theorem; see [Ser18, Theorem 1.5 of Chapter 2] for a proof.

Lemma 5.1 (Tychonoff Uniquenesss Theorem)**.** *Let $T > 0$ and $M > 0$. Let v be a measurable function on $[0,T] \times \mathbb{R}^d$ such that for all $(t,x) \in [0,T] \times \mathbb{R}^d$,*

$$|v(t,x)| \leqslant M \exp\big(M|x|^2\big),$$

and for all[10] *$\phi \in C_c^\infty((-1,T) \times \mathbb{R}^d)$,*

$$\int_0^T \int_{\mathbb{R}^d} v(\partial_t \phi + \Delta\phi)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Then $v = 0$.

Lemma 5.2. *Let $f \in L^\infty([0,\infty) \times \mathbb{R}^{\mathrm{d}}) \cap \bigcap_{T>0} L^2([0,T] \times \mathbb{R}^{\mathrm{d}})$. Suppose that u is bounded and measurable and solves*

$$\begin{cases} \partial_t u = \frac{1}{2}\Delta u + f, \\ u(0,x) = 0. \end{cases}$$

in $[0,\infty) \times \mathbb{R}^{\mathrm{d}}$.

Then for all $T > 0$,

$$\|u\|_{\infty,\,[0,T]\times\mathbb{R}^{\mathrm{d}}} \leqslant T\|f\|_{\infty,\,[0,T]\times\mathbb{R}^{\mathrm{d}}}.$$

Proof. Fix $T > 0$. As $f \in L^2([0,T] \times \mathbb{R}^{\mathrm{d}})$, there exist $f_m \in C_c^\infty((0,T) \times \mathbb{R}^{\mathrm{d}})$ such that $f_m \to f$ in $L^2([0,T]\times\mathbb{R}^{\mathrm{d}})$. Let Φ be the heat kernel on $[0,\infty)\times\mathbb{R}^{\mathrm{d}}$, appropriately rescaled so that it in fact solves $\partial_t\Phi - \frac{1}{2}\Delta\Phi = 0$. Let

$$\begin{aligned} v(t,x) &= \int_0^t \int_{\mathbb{R}^{\mathrm{d}}} \Phi(t-\tau, x-y) f(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau, \\ v_m(t,x) &= \int_0^t \int_{\mathbb{R}^{\mathrm{d}}} \Phi(t-\tau, x-y) f_m(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \end{aligned}$$

for $t \geqslant 0$, $x \in \mathbb{R}^{\mathrm{d}}$. Note that $v_m \to v$ in $L^2([0,T] \times \mathbb{R}^{\mathrm{d}})$.

Then v_m satisfies the heat equation with source

$$\begin{cases} \partial_t v_m = \dfrac{1}{2}\Delta v_m + f_m \\ v_m(0,x) = 0 \end{cases}$$

in $(0,\infty)\times\mathbb{R}^{\mathrm{d}}$. (See [Eva10, Theorem 2 of Section 2.3]). In particular it is a distributional solution: for all $\phi \in C_c^\infty((-1,T) \times \mathbb{R}^{\mathrm{d}})$,

$$\int_0^T \int_{\mathbb{R}^{\mathrm{d}}} v_m \left(\partial_t\phi + \frac{1}{2}\Delta\phi\right) \mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_{\mathbb{R}^{\mathrm{d}}} f_m \phi\,\mathrm{d}x\,\mathrm{d}t.$$

Now $f_m \to f$ and $v_m \to v$ in $L^2([0,T] \times \mathbb{R}^{\mathrm{d}})$, so letting $m \to \infty$ gives

$$\int_0^T \int_{\mathbb{R}^{\mathrm{d}}} v \left(\partial_t\phi + \frac{1}{2}\Delta\phi\right) \mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_{\mathbb{R}^{\mathrm{d}}} f \phi\,\mathrm{d}x\,\mathrm{d}t.$$

[10] That ϕ is defined down to time -1 is just so that it need not be zero at time zero despite having compact support.

Except of course u satisfies the same equation, so $u - v$ is a weak solution to the homogenous equation. Now f is bounded so v is bounded, and by assumption u is bounded. So we may apply the Tychonoff Uniqueness Theorem (Lemma 5.1) to $u - v$ to deduce that in fact $u = v$.

Finally, using the fact that $\int_{\mathbb{R}^d} \Phi \,\mathrm{d}x = 1$, it is clear that $\|v\|_{\infty,\,[0,T]\times\mathbb{R}^d} \leqslant T\|f\|_{\infty,\,[0,T]\times\mathbb{R}^d}$, giving the desired result. □

Lemma 5.3. *Let $p_0 \in C_c(\mathbb{R}^d; [0,1])$ and let $v \in C^2_1([0,\infty) \times \mathbb{R}^d; [0,1])$ solve the Fisher–KPP equation*

$$\begin{cases} \partial_t v = \frac{1}{2}\Delta v + \eta(1-v)(1-(1-v)^{k-1}), \\ v(0,x) = p_0(x). \end{cases}$$

Then

$$v \in L^\infty([0,\infty) \times \mathbb{R}^d) \cap \bigcap_{T>0} L^2([0,T] \times \mathbb{R}^d)$$

and

$$(1-v)(1-(1-v)^{k-1}) \in L^\infty([0,\infty) \times \mathbb{R}^d) \cap \bigcap_{T>0} L^2([0,T] \times \mathbb{R}^d).$$

Proof. As v maps into $[0,1]$ the L^∞ bound is trivial.

Fix $T > 0$. We note that $(1-v)(1-(1-v)^{k-1}) = vp_{k-1}(v)$ for some polynomial p_{k-1} of degree $k-1$. Furthermore, v maps into $[0,1]$, so $p_{k-1}(v)$ is bounded by some constant c_0 depending only on k. So

$$(1-v)(1-(1-v)^{k-1}) = vp_{k-1}(v) \leqslant c_0 v.$$

Now using that both v and $p_{k-1}(v)$ are bounded, by the Feynman–Kac formula,

$$\begin{aligned} v(t,x) &= \mathbb{E}_x\left[p_0(B_t)\exp\left(\eta\int_0^t p_{k-1}(v(t-\tau, B_\tau))\,\mathrm{d}\tau\right)\right] \\ &\leqslant \mathbb{E}_x\left[p_0(B_t)\exp\left(\eta t c_0\right)\right] \\ &\leqslant \exp\left(\eta T c_0\right)\mathbb{E}_x\left[p_0(B_t)\right]. \end{aligned}$$

Recognising that $\mathbb{E}_x[p_0(B_t)] \in L^2([0,T] \times \mathbb{R}^d)$ as it is the bounded solution to the heat equation with compactly supported initial data p_0 then gives the result. □

5.1 Small η

First we consider sending $\eta \to 0$, and allow k to do whatever it pleases. This means that the nonlinearity tends to zero uniformly in v, so we will deduce uniform convergence to the heat equation.

Proposition 5.4. *Let $p_0 \in C_c(\mathbb{R}^d; [0,1])$. For $m \in \mathbb{N}$, let (k_m) be any sequence on $\{2,3,4,\ldots\}$, and let (η_m) be a positive sequence such that $\eta_m \to 0$ as $m \to \infty$. Let $v_m \in C^2_1([0,\infty) \times \mathbb{R}^d; [0,1])$ solve the Fisher–KPP equation*

$$\begin{cases} \partial_t v_m = \frac{1}{2}\Delta v_m + \eta_m(1-v_m)(1-(1-v_m)^{k_m-1}), \\ v_m(0,x) = p_0(x), \end{cases}$$

and let $v \in C^\infty((0,\infty)\times\mathbb{R}^d)\cap C([0,\infty)\times\mathbb{R}^d)$ *solve the heat equation*[11]

$$\begin{cases}\partial_t v = \frac{1}{2}\Delta v,\\ v(0,x) = p_0(x).\end{cases}$$

Then for all $T > 0$,

$$\|v - v_m\|_{\infty,\,[0,T]\times\mathbb{R}^d} = \mathcal{O}(\eta_m)$$

That is, $v_m \to v$ *in* $L^\infty([0,T]\times\mathbb{R}^d)$, *which implies that* $v_m \overset{*}{\rightharpoonup} v$ *in* $L^\infty([0,\infty)\times\mathbb{R}^d)$.

Proof. Fix $T > 0$. Let $f_m = \eta_m(1-v_m)(1-(1-v_m)^{k_m-1})$. So $f_m \in L^\infty([0,T]\times\mathbb{R}^d)\cap \bigcap_{T>0} L^2([0,T]\times\mathbb{R}^d)$ by Lemma 5.3. Now $v_m - v$ satisfies

$$\begin{cases}\partial_t(v_m - v) = \frac{1}{2}\Delta(v_m - v) + f_m,\\ v_m(0,x) - v(0,x) = 0,\end{cases}$$

so by Lemma 5.2,

$$\|v_m - v\|_{\infty,\,[0,T]\times\mathbb{R}^d} \leqslant T\|f_m\|_{\infty,\,[0,T]\times\mathbb{R}^d} = \mathcal{O}(\eta_m).$$

It is clear that this immediately implies that $v_m \to v$ in $[0,\infty)\times\mathbb{R}^d$ in the sense of distributions: just pick T large enough that $[0,T]\times\mathbb{R}^d$ contains the support of the chosen test function. As v_m are uniformly bounded in $L^\infty([0,\infty)\times\mathbb{R}^d)$, then it is elementary that in fact the convergence is weak* as well: let $g \in L^1([0,\infty)\times\mathbb{R}^d)$, and let $g_n \in C_c^\infty((0,\infty)\times\mathbb{R}^d$ be such that $g_n \to g$ in $L^1([0,\infty)\times\mathbb{R}^d)$. Then

$$\begin{aligned}&\left|\int_0^\infty\int_{\mathbb{R}^d} vg\,\mathrm{d}x\,\mathrm{d}t - \int_0^\infty\int_{\mathbb{R}^d} v_m g\,\mathrm{d}x\,\mathrm{d}t\right|\\ &\leqslant \|vg - vg_n\|_{1,\,[0,\infty)\times\mathbb{R}^d} + \|vg_n - v_m g_n\|_{1,\,[0,\infty)\times\mathbb{R}^d} + \|v_m g_n - v_m g\|_{1,\,[0,\infty)\times\mathbb{R}^d}\\ &\leqslant 2\|g - g_n\|_{1,\,[0,\infty)\times\mathbb{R}^d} + \|vg_n - v_m g_n\|_{1,\,[0,\infty)\times\mathbb{R}^d}.\end{aligned}$$

and now pick n large enough that the first term is small, and then pick m large enough that the middle term is small. □

5.2 Constant η, large k

Lemma 5.5 (Weak Maximum Principle)**.** *Let* $c \in \mathbb{R}$. *Let* $u \in C_1^2([0,\infty)\times\mathbb{R}^d;\mathbb{R})$ *be bounded and such that*

$$\partial_t u - \frac{1}{2}\Delta u + cu \leqslant 0,$$

in $[0,\infty)\times\mathbb{R}^d$.

Suppose also that $u \leqslant 0$ *on* $\{0\}\times\mathbb{R}^d$.

Then

$$\sup_{\substack{x\in\mathbb{R}^d\\ t\in[0,\infty)}} u(t,x) \leqslant 0.$$

[11] See [Eva10, Theorem 1 of Section 2.3] for existence of such a u.

Remark 5.6. In particular note that no assumption is made on the sign of c. Note also by continuity and decay of u that both supremums are in fact maximums. Finally, note that it should be possible to strengthen the result to those u merely satisfying a growth condition of the form $u(t,x) \leqslant A\exp\big(a|x|^2\big)$ on compact time intervals, but this is an unnecessary complication for us. (See [Eva10, Theorem 6 of Section 2.3].)

Proof of Lemma 5.5. Let $\varepsilon > 0$ and let

$$v_\varepsilon(t,x) = \mathrm{e}^{ct}u(t,x) - \varepsilon(\mathrm{d}t + |x|^2)$$

Then

$$\partial_t v_\varepsilon - \tfrac{1}{2}\Delta v_\varepsilon \leqslant 0$$

in $[0,\infty)\times\mathbb{R}^{\mathrm{d}}$.

Fix $T>0$ and $R>0$. Let $\Omega = [0,T]\times B(0,R)$. Let u be bounded by C. Now

$$v_\varepsilon \leqslant \mathrm{e}^{cT}C - \varepsilon R^2$$

on $[0,T]\times\partial B(0,R)$, and

$$v_\varepsilon \leqslant 0.$$

on $\{0\}\times B(0,R)$. So by the weak maximum principle for the heat equation (see [Eva10, Theorem 8 of Section 7.1]),

$$\sup_{\substack{x\in B(0,R)\\ t\in[0,T]}} v_\varepsilon(t,x) \leqslant \max\left\{\mathrm{e}^{cT}C - \varepsilon R^2,\, 0\right\}.$$

Let $R\to\infty$ to deduce that for $x\in\mathbb{R}^{\mathrm{d}}$ and $t\in[0,T]$,

$$v_\varepsilon(t,x) \leqslant \sup_{\substack{x\in\mathbb{R}^{\mathrm{d}}\\ t\in[0,T]}} v_\varepsilon(t,x) \leqslant 0.$$

And so

$$\begin{aligned}\sup_{\substack{x\in\mathbb{R}^{\mathrm{d}}\\ t\in[0,T]}} u(t,x) &\leqslant \sup_{t\in[0,T]} \mathrm{e}^{-ct}\cdot \sup_{\substack{x\in\mathbb{R}^{\mathrm{d}}\\ t\in[0,T]}} \mathrm{e}^{ct}u(t,x)\\ &= \max\{1,\mathrm{e}^{-cT}\}\cdot \sup_{\substack{x\in\mathbb{R}^{\mathrm{d}}\\ t\in[0,T]}} \lim_{\varepsilon\to 0} v_\varepsilon(t,x)\\ &\leqslant 0.\end{aligned}$$

Finally let $T\to\infty$ to deduce the result. □

Lemma 5.7. *Let $p_0\in C_c(\mathbb{R}^{\mathrm{d}};[0,1])$. Let $\eta>0$. For $m\in\mathbb{N}$, let (k_m) be a monotonically nondecreasing sequence on $\{2,3,4,\ldots\}$. Let $v_m\in C_1^2([0,\infty)\times\mathbb{R}^{\mathrm{d}};[0,1])$ solve the Fisher–KPP equation*

$$\begin{cases}\partial_t v_m = \tfrac{1}{2}\Delta v_m + \eta(1-v_m)(1-(1-v_m)^{k_m-1}),\\ v_m(0,x) = p_0(x).\end{cases}$$

Then (v_m) is pointwise monotonically nondecreasing.

Proof. For ease of notation, let $u_m = 1 - v_m$, so that u_m solves

$$\begin{cases} \partial_t u_m = \frac{1}{2}\Delta u_m + \eta(u_m^{k_m} - u_m), \\ u_m(0,x) = 1 - p_0(x). \end{cases}$$

We will show that (u_m) is pointwise monotonically nonincreasing.

Fix $m, n \in \mathbb{N}$ such that $n \leqslant m$, so $k_n \leqslant k_m$. As u_m maps into $[0,1]$, this means that $u_m^{k_m} \leqslant u_m^{k_n}$, and thus that

$$\begin{aligned} u_m^{k_m} - u_n^{k_n} &\leqslant u_m^{k_n} - u_n^{k_n} \\ &= (u_m - u_n)\sum_{j=0}^{k_n - 1} u_m^{k_n - j - 1} u_n^j \\ &\leqslant k_n(u_m - u_n). \end{aligned}$$

And hence

$$\begin{aligned} \partial_t(u_m - u_n) - \frac{1}{2}\Delta(u_m - u_n) &= \eta(u_m^{k_m} - u_n^{k_n} - u_m + u_n) \\ &\leqslant \eta(k_n - 1)(u_m - u_n). \end{aligned}$$

Let $c_1 = -\eta(k_n - 1)$. Then $w = u_m - u_n$ is bounded and satisfies

$$\begin{cases} \partial_t w - \frac{1}{2}\Delta w + c_1 w \leqslant 0, \\ w(0,x) = 0. \end{cases}$$

So Lemma 5.5 implies that $w \leqslant 0$ in $[0,\infty) \times \mathbb{R}^{\mathrm{d}}$, which gives the result. □

Proposition 5.8. *Let $p_0 \in C_c(\mathbb{R}^{\mathrm{d}}; [0,1])$ be nontrivial*[12]*. Let $\eta > 0$. For $m \in \mathbb{N}$, let (k_m) be a sequence on $\{2,3,4,\ldots\}$ tending to infinity. Let $v_m \in C_1^2([0,\infty) \times \mathbb{R}^{\mathrm{d}}; [0,1])$ solve the Fisher–KPP equation*

$$\begin{cases} \partial_t v_m = \frac{1}{2}\Delta v_m + \eta(1 - v_m)(1 - (1 - v_m)^{k_m - 1}), \\ v_m(0,x) = p_0(x). \end{cases}$$

Then there exists $v \in C^\infty((0,\infty) \times \mathbb{R}^{\mathrm{d}}; [0,1]) \cap C([0,\infty) \times \mathbb{R}^{\mathrm{d}}; [0,1])$ solving the equation

$$\begin{cases} \partial_t v = \frac{1}{2}\Delta v + \eta(1 - v), \\ v(0,x) = p_0(x). \end{cases} \tag{5.1}$$

such that for all $T > 0$, $v_m \to v$ in $L^\infty([0,T] \times \mathbb{R}^{\mathrm{d}})$, and $v_m \overset{*}{\rightharpoonup} v$ in $L^\infty([0,\infty) \times \mathbb{R}^{\mathrm{d}})$.

Proof. First we show uniqueness of v. Suppose there exists $v^{(1)}$, $v^{(2)}$ solving equation (5.1). Then $\psi = v^{(1)} - v^{(2)}$ solves

$$\begin{cases} \partial_t \psi = \frac{1}{2}\Delta\psi - \eta\psi, \\ \psi(0,x) = 0. \end{cases}$$

[12] This assumption isn't absolutely necessary, but the convergence is no longer to the solution of equation (5.1). In the following proof we prove that a particular term tends to zero, and in this edge case that will no longer hold. If we wished we could allow p_0 to be trivial and place an extra $-\eta \mathbb{1}_{\{0\}}(v)$ term in equation (5.1).

Now $v^{(1)}$ and $v^{(2)}$ are bounded so ψ is bounded, so Lemma 5.5 implies that $v^{(1)} \leqslant v^{(2)}$. Reversing the roles of $v^{(1)}$ and $v^{(2)}$ gives uniqueness.

Next, consider any subsequence of (v_m). As $k_m \to \infty$, we may pick a further subsequence, call it (v_{m_j}), such that k_{m_j} is monotonically increasing. Suppose we can show our Proposition for this subsequence, namely that there exists some v satisfying equation (5.1) such that $v_{m_j} \to v$ in $L^\infty([0,T] \times \mathbb{R}^d)$ and $v_{m_j} \overset{*}{\rightharpoonup} v$ in $L^\infty([0,\infty) \times \mathbb{R}^d)$, where in principle the limit v may depend on the choices of subsequences. But we have just demonstrated that we have uniqueness of solution to equation (5.1), and so in fact every such subsequence much converge to the same limit. Now a sequence converges if and only if all subsequences have a subsequence converging to the same value, and so (v_m) itself must converge to v in these ways. So without loss of generality, (k_m) may be assumed to be monotonically increasing.

With this assumption, Lemma 5.7 implies that (v_m) is pointwise monotonically non-decreasing. Furthermore, each v_m is bounded (as they map into $[0,1]$), and so in fact (v_m) is pointwise convergent, say to v.

Now for every $\phi \in C_c^\infty((-1,\infty) \times \mathbb{R}^d)$,

$$\int_0^\infty \int_{\mathbb{R}^d} v_m \left(\partial_t \phi + \frac{1}{2} \Delta \phi \right) \mathrm{d}x \, \mathrm{d}t \to \int_0^\infty \int_{\mathbb{R}^d} v \left(\partial_t \phi + \frac{1}{2} \Delta \phi \right) \mathrm{d}x \, \mathrm{d}t \tag{5.2}$$

by monotone convergence.

But also,

$$\begin{aligned}
&\int_{\mathbb{R}^d} p_0(x)\phi(0,x) \, \mathrm{d}x + \int_0^\infty \int_{\mathbb{R}^d} v_m \left(\partial_t \phi + \frac{1}{2} \Delta \phi \right) \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^\infty \int_{\mathbb{R}^d} \left(-\partial_t v_m + \frac{1}{2} \Delta v_m \right) \phi \, \mathrm{d}x \, \mathrm{d}t \\
&= -\int_0^\infty \int_{\mathbb{R}^d} \eta(1 - v_m)(1 - (1 - v_m)^{k_m - 1}) \phi \, \mathrm{d}x \, \mathrm{d}t \\
&= -\int_0^\infty \int_{\mathbb{R}^d} \eta(1 - v_m)\phi - \eta(1 - v_m)^{k_m} \phi \, \mathrm{d}x \, \mathrm{d}t \\
&\to -\int_0^\infty \int_{\mathbb{R}^d} \eta(1 - v)\phi \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned} \tag{5.3}$$

This convergence needs a little explanation. The first term converges by monotone convergence again. As for the second term, we begin by noting that $\partial_t v_1 - \frac{1}{2}\Delta v_1 = \eta(1 - v_1)(1 - (1 - v_1)^{k_1 - 1}) \geqslant 0$, so by the strong maximum (minimum) principle,[13] if v_1 ever equalled its minimum value of zero at some point of $(0,\infty) \times \mathbb{R}^d$, then it must be the constant zero function. As $p_0 \not\equiv 0$ then this is a contradiction. Thus $v_1 > 0$ in $(0,\infty) \times \mathbb{R}^d$, and so

$$1 - v_1 < 1. \tag{5.4}$$

Thus

$$\begin{aligned}
\left| \int_0^\infty \int_{\mathbb{R}^d} \eta(1 - v_m)^{k_m} \phi \, \mathrm{d}x \, \mathrm{d}t \right| &\leqslant \int_0^\infty \int_{\mathbb{R}^d} \eta(1 - v_m)^{k_m} |\phi| \, \mathrm{d}x \, \mathrm{d}t \\
&\leqslant \int_0^\infty \int_{\mathbb{R}^d} \eta(1 - v_1)^{k_m} |\phi| \, \mathrm{d}x \, \mathrm{d}t \\
&\to 0
\end{aligned}$$

[13] See [Eva10, Theorem 11 of Section 7.1]

where the second inequality is by monotonicity of (v_m), and the convergence is by monotonicity of (k_m) and equation 5.4.

And so equations (5.2) and (5.3) together imply that v is a distributional solution to equation (5.1). (In particular including the initial condition.) So let $w(t,x) = v(t,x)\mathrm{e}^{\eta t} + 1 - \mathrm{e}^{\eta t}$. Then w is a distributional solution to the heat equation with initial condition p_0. But we know that there exists a smooth bounded solution $\widetilde{w}$ to the heat equation with initial condition p_0. As the v_m map into $[0,1]$ and converge pointwise to v then v must map into $[0,1]$ as well. Hence w is bounded on compact time intervals, so we may apply the Tychonoff Uniqueness Theorem (Lemma 5.1) to $w - \widetilde{w}$ to deduce that in fact $w = \widetilde{w}$, and so in particular w is smooth. Thus v is smooth as well. Similarly, $\widetilde{w}$ is continuous up to the initial time $\{t = 0\}$, so v is as well.

Now for fixed m, n, let $w(t,x) = (v_m(t,x) - v_n(t,x))\mathrm{e}^{\eta t}$. Then w satisfies

$$\begin{cases} \partial_t w = \frac{1}{2}\Delta w + \eta \mathrm{e}^{\eta t}\left[(1 - v_n)^{k_n} - (1 - v_m)^{k_m}\right], \\ w(0,x) = 0. \end{cases}$$

Now

$$\begin{aligned} &(1 - v_n)^{k_n} - (1 - v_m)^{k_m} \\ &= (1 - v_m)(1 - (1 - v_m)^{k_m - 1}) - (1 - v_n)(1 - (1 - v_n)^{k_n - 1}) + v_m - v_n \end{aligned}$$

which belongs to $L^\infty([0,T] \times \mathbb{R}^{\mathrm{d}}) \cap \bigcap_{T>0} L^2([0,T] \times \mathbb{R}^{\mathrm{d}})$ by Lemma 5.3. So Lemma 5.2 implies that

$$\begin{aligned} \|w\|_{\infty,\,[0,T]\times\mathbb{R}^{\mathrm{d}}} &\leqslant T\left\|\eta \mathrm{e}^{\eta t}\left[(1 - v_n)^{k_n} - (1 - v_m)^{k_m}\right]\right\|_{\infty,\,[0,T]\times\mathbb{R}^{\mathrm{d}}}. \\ &\leqslant T\eta \mathrm{e}^{\eta T}\left[\left\|(1 - v_1)^{k_n}\right\|_{\infty,\,[0,T]\times\mathbb{R}^{\mathrm{d}}} + \left\|(1 - v_1)^{k_m}\right\|_{\infty,\,[0,T]\times\mathbb{R}^{\mathrm{d}}}\right] \\ &\to 0 \end{aligned}$$

as $m, n \to \infty$, with convergence because $1 - v_1$ is uniformly bounded away from one on compact time intervals, as it satisfies equation 5.4, is continuous, and by Lemma 5.3 belongs to $L^2([0,T] \times \mathbb{R}^{\mathrm{d}})$.

This implies that (v_m) is $L^\infty([0,T] \times \mathbb{R}^{\mathrm{d}})$-Cauchy, so the result now follows in the same was as in the end of the proof of Proposition 5.4. □

Remark 5.9. We see from the above proof that in fact it is possible to characterise the limit function v as $v(t,x) = w(t,x)\mathrm{e}^{-\eta t} + 1 - \mathrm{e}^{-\eta t}$, where w solves the heat equation with initial condition p_0. This is also a way in which uniqueness of v could be proved.

Remark 5.10. We note, as in Remark 2.2, that we expect that we expect the results of this subsection to hold for random k as well, provided that the meaning of k being both random and large is suitably interpreted. For example, it is sufficient for k_m to have probability generating function $\sum_{i=l_m}^{N_m} k_m^{(i)} x^i$, with $l_m, N_m \to \infty$. The corresponding notion of monotonicity that is recovered (without loss of generality) is that for all $m, n, N \in \mathbb{N}$ such that $n \leqslant m$ that $\sum_{i=0}^{N} \left(k_m^{(i)} - k_n^{(i)}\right) \leqslant 0$, where $k_m^{(i)} = 0$ if $i < l_m$.

Remark 5.11. In both the small η and the constant η, large k case, the key fact that we are exploiting is that the nonlinear term '$\eta(1-v)^k$' converges uniformly to zero. As such — in a similar fashion to the next remark — it should be possible to extend to a 'convergent η, large k' case.

Remark 5.12. We note that it should be possible to extend our above results to sequences of functions with varying initial data, provided it converges uniformly. This means that we will get nonzero initial conditions in our comparisons between solutions. The proof of Lemma 5.2 may be extended to cover a nonzero initial condition, in which case the L^∞ bound of this initial condition will appear on the right hand side. Proposition 5.8 then needs some adaptation: varying initial conditions mean that monotonicity need not hold. Fortunately, we only use monotonicity as it is convenient, but it is not integral to the proof — bounded convergence may be substituted. Some care is necessary to bound the nonlinearity away from one (to ensure that the limit function does not have a '$\mathbb{1}_{\{0\}}(v)$' term in its equation): this is done by constructing an initial condition function p eventually bounding the sequence of initial condition functions below, say from m onwards, and then comparing to the solution of

$$\begin{cases} \partial_t v = \frac{1}{2}\Delta v + \eta(1-v)(1-(1-v)^{k_m-1}), \\ v(0,x) = p(x). \end{cases}$$

It is clear that we may now combine the results of this section and our last section to deduce L^∞_{loc} convergence of w^n to solutions of the heat equation or the Fisher–KPP equation: Fix m sufficiently large, then apply Corollary 4.5 (or Theorem 4.4) with $\eta = \eta_m$, $k = k_m$, and then apply Proposition 5.4 or Proposition 5.8. This gives:

Theorem 5.13. *Assume as in Corollary 4.5. Furthermore, allow u, s, R, k, now denoted u_m, s_m, R_m, k_m, to be sequences in $m \in \mathbb{N}$, and let $\eta_m = u_m s_m R_m^{\mathrm{d}} V_1$.*

(i) If $\eta_m \to 0$ as $m \to \infty$, then let $v \in C^\infty((0,\infty) \times \mathbb{R}^{\mathrm{d}}; [0,1]) \cap C([0,\infty) \times \mathbb{R}^{\mathrm{d}}; [0,1])$ solve the heat equation

$$\begin{cases} \partial_t v = \frac{1}{2}\Delta v, \\ v(0,x) = p_0(x). \end{cases}$$

Then for all $t \in [0, T\log n]$ and $x \in \mathbb{R}^d$,

$$\left|\mathbb{E}_{p_0}\left[w_t^n(x)\right] - v(\sigma^2 t, x)\right| = \mathcal{O}\left(n^{-\alpha} + \eta_m\right).$$

(ii) Assume also that p_0 is nontrivial. If $\eta_m = \eta$ is constant and $k_m \to \infty$ as $m \to \infty$, then there exists $v \in C^\infty((0,\infty) \times \mathbb{R}^{\mathrm{d}}; [0,1]) \cap C([0,\infty) \times \mathbb{R}^{\mathrm{d}}; [0,1])$ solving the equation

$$\begin{cases} \partial_t v = \frac{1}{2}\Delta v + \eta(1-v), \\ v(0,x) = p_0(x), \end{cases}$$

such that for all $t \in [0, T\log n]$ and $x \in \mathbb{R}^d$,

$$\left|\mathbb{E}_{p_0}\left[w_t^n(x)\right] - v(\sigma^2 t, x)\right| \to 0$$

uniformly in t and x as both $n, m \to \infty$. (The order of limits does not matter.)

6 Alternate Scaling Limits

We now move on to considering the scenario that η and k depend explicitly on n. Much of our initial set-up will mimic that of Section 3. We will have to be careful to ensure that the result of our scaling limit remains true: once we have that, then we may apply the results of Section 5 to deduce the overall convergence.

For our first scaling limit, in which we demand that η be small, we find that we converge to the heat equation, see Theorem 6.2. This is clearly not modelling population expansion in the manner that we wish; if anything it is merely representing population redistribution! Thus we see that this model is unsuitable. The reason for this behaviour is that in demanding that η become small, we do so by letting the reproduction parameter s, which determines the proportion of reproduction events which are expansive, become small. We let s become small — rather than say, letting u become small, which would also give the desired affect of letting η become small — as doing so requires the fewest modifications to our arguments in Section 3. Attempting to reproduce this case by scaling in some other way merits further investigation.

In the second scaling limit, we try a little harder, and ask that η be constant whilst k grows large, see Theorem 6.5. Here we find that we get the opposite problem: all of space finds itself with a growing amount of population, subject to a perturbation describing the effect of the initial data: the population density is asymptotically homogenous in each of space and time. See Remark 5.9.

Starting from the Fisher–KPP equation, it is clear that there is little in the way of other viable limits to consider. Thus we see that if we hope to model population expansion in a manner similar to that which is done here, then we have perhaps two choices: the first is accept the Fisher–KPP equation itself (the result of our simpler scaling limit in Section 4) as modelling population expansion. This is not such a bad choice; the equation has already been used to model population expansion, see [Ske51] or perhaps [VOAE14]. The second is to perhaps attempt to reproduce the small η case in some other way, as discussed above.

6.1 Small η

Let $\mathbb{d}, \beta, u, s, R$ be as in Section 3. For $n \in \mathbb{N}$, let (k_n) be any sequence on $\{2, 3, 4, \ldots\}$ such that $k_n = \mathcal{O}(\log n)$. In particular pick k such that[14] $k_n \leqslant k \log n$. Now for $n \in \mathbb{N}$ let

$$s_n = \frac{s}{n^{2\beta}(k \log n - 1)}$$

Let (u_n) and (R_n) be as in Section 3.

Let Π^n be a Poisson point process on $[0, \infty) \times \mathbb{R}^{\mathbb{d}}$ with intensity measure $n\mathrm{d}t \otimes n^{\beta}\mathrm{d}x$ as in Section 3. For each $n \in \mathbb{N}$, let w^n be an SLFVE and Ξ^n be an SLFVE dual, in each case driven by Π^n, with parameters (u_n, s_n, R_n, k_n).

Let m_n be the jump intensity of a single lineage in Ξ^n. It is the same as in Section 3, so the total rate of jumps is once again

$$un^{2\beta}R^{\mathbb{d}}V_1.$$

[14] Technically speaking we're playing a little fast and loose for small n: this inequality cannot hold for $n = 1$, as then $2 \leqslant k_1 \leqslant k \log 1 = 0$. Nonetheless we keep this description for clarity of notation, as we're only interested in the tail anyway.

Now $1 - s_n$ of the jumps will be from neutral events, and s_n of the jumps will be from expansive events. Thus the lineage is affected by expansive events at rate η_n, where

$$\eta_n = un^{2\beta} R^{\mathrm{d}} V_1 s_n = \frac{usR^{\mathrm{d}} V_1}{k \log n - 1}$$

which we note is no longer independent of n. Let

$$\lambda = \eta_n (k \log n - 1) = usR^{\mathrm{d}} V_1$$

which remains independent of n. Note that $\lambda \geqslant \eta_n$ as $k \log n - 1 \geqslant k_n - 1 \geqslant 1$.

Let σ^2 be as in Section 3.

The technical conditions are easier to handle this time around. Let $\alpha > 0$ and $T > 0$, and pick $b > 0$ *large* enough — note that before we had to pick it sufficiently small — that[15]

$$T\lambda < b \tag{6.1}$$

$$b + b \log\left(\frac{eT\lambda}{b}\right) + \alpha + T\lambda < 0. \tag{6.2}$$

Finally let

$$T_n = T \log\log n \qquad \text{and} \qquad b_n = b \log \log n.$$

We now move on to investigating how each our of our results changes under this scaling; in all cases the arguments involved remain the same so we shall merely summarise the key points.

(i) The maximal number of expansive events, as in Lemma 3.1, is now

$$\mathbb{P}\left[\mathrm{Expan}_{T_n}(\Xi^n) > b_n\right] = o((\log n)^{-\alpha - T\lambda}).$$

To see this, let $Z_n \sim \mathrm{Poisson}(T_n \eta_n)$, so that then by equation (6.1) we may apply the Chernoff bound to give that

$$\mathbb{P}[Z_n > b_n] \leqslant \left(\frac{eT_n\eta_n}{b_n}\right)^{b_n} = \left(\frac{eT\lambda}{b(k\log n - 1)}\right)^{b \log\log n}.$$

Now equation (6.2) implies that

$$\left[b \log\left(\frac{k \log n}{k \log n - 1}\right) + b \log\left(\frac{eT\lambda}{b}\right) + \alpha + T\lambda\right] \log\log n \to -\infty$$

Thus bounding the maximal number of expansive events by

$$k_n^{b_n} \cdot \mathbb{P}[Z_n > b_n] \leqslant (k \log n)^{b \log\log n} \cdot \left(\frac{eT\lambda}{b(k \log n - 1)}\right)^{b \log \log n} = o((\log n)^{-\alpha - T\lambda}).$$

[15] If we wanted to be consistent with the previous section, then we could instead take $b > 0$ arbitrary and impose that $T > 0$ be sufficiently small that the condition holds. But in some sense T is 'more important' than b (for example, only T appears in Corollary 4.5), so it makes sense to grant it the greater freedom.

(ii) The probability of having simultaneously marked individuals, as in Proposition 3.2, is now

$$\mathbb{P}\left[\mathrm{Simul}_{T_n}(\Xi^n)\right] = o((\log n)^{-\alpha}).$$

To see this, it suffices to show that

$$k_n^{2b_n} \cdot n^{4\beta-1} \cdot \log n = o((\log n)^{-\alpha})$$

which is easy, recalling that $4\beta - 1 < 0$.

(iii) The expected number of lineages, conditioned on $\mathrm{NoSimul}_{T_n}(\Xi^n)$, as in Lemma 3.3, is now bounded by

$$\mathbb{E}\left[\mathrm{Lin}_{T_n}(\Xi^n) \middle| \mathrm{NoSimul}_{T_n}(\Xi^n)\right] = \exp(\eta_n(k_n - 1)T_n) \leqslant \exp(\lambda T_n) = (\log n)^{T\lambda}.$$

This also bounds the expected number of individuals, as in Corollary 3.4.

(iv) Applying these changes to Proposition 3.6, we find that the approximation to branching Brownian motion now occurs at rate $o((\log n)^{-\alpha})$.

(v) Adjusting the definition of D_2 in Proposition 3.8 to $\{N_t^n \leqslant (\log n)^{T\lambda+\alpha}\}$ gives our overall convergence at rate

$$\mathcal{O}((\log n)^{-\alpha} + (\log n)^{T\lambda+\alpha} \cdot \omega(4n^{-\beta/6}b_n)).$$

And so we can now bring these results together to state an equivalent to Theorem 4.4 describing this alternate scaling:

Proposition 6.1. *Let $\mathbb{d} \in \mathbb{N}$. Let $\beta \in (0, 1/4)$. Let $\alpha > 0$. Let $u \in (0, 1)$, $s \in (0, 1)$, $k > 0$, $R > 0$ and $T > 0$. Let V_1 be the volume of the ball $B(0, R) \subseteq \mathbb{R}^{\mathbb{d}}$. Let $\lambda = usR^{\mathbb{d}}V_1$. For $n \in \mathbb{N}$, let (k_n) be any sequence on $\{2, 3, 4, \ldots\}$ such that $k_n \leqslant k \log n$.*

For each $n \in \mathbb{N}$, let

$$u_n = \frac{u}{n^{1-2\beta}} \qquad \text{and} \qquad s_n = \frac{s}{n^{2\beta}(k\log n - 1)} \qquad \text{and} \qquad R_n = \frac{R}{n^\beta}$$

$$\text{and} \qquad T_n = T \log\log n \qquad \text{and} \qquad \eta_n = \frac{\lambda}{k\log n - 1}.$$

Also let

$$\sigma^2 = \frac{uR^{\mathbb{d}}V_1}{\mathbb{d}}.$$

Let Π^n be a Poisson point process on $[0, \infty) \times \mathbb{R}^{\mathbb{d}}$ with intensity measure $n\mathrm{d}t \otimes n^\beta \mathrm{d}x$. Let $p_0 \in C_c(\mathbb{R}^{\mathbb{d}}; [0, 1])$ have modulus of continuity ω.

For each $n \in \mathbb{N}$, let w^n be an SLFVE driven by Π^n with parameters (u_n, s_n, R_n, k_n), and let $v_n \in C_1^2([0, \infty) \times \mathbb{R}^{\mathbb{d}}; [0, 1])$ satisfy the Fisher–KPP equation

$$\begin{cases} \partial_t v_n = \frac{1}{2}\Delta v_n + \eta_n(1 - v_n)(1 - (1 - v_n)^{k_n - 1}), \\ v_n(0, x) = p_0(x). \end{cases}$$

Then there exists $b > 0$ sufficiently large that for all $t \in [0, T_n]$ and $x \in \mathbb{R}^{\mathbb{d}}$,

$$\left|\mathbb{E}_{p_0}\left[w_t^n(x)\right] - v_n(\sigma^2 t, x)\right| = \mathcal{O}((\log n)^{-\alpha} + (\log n)^{T\lambda+\alpha} \cdot \omega(4n^{-\beta/6} b \log\log n)).$$

Which we may now combine with Proposition 5.4 (which we note in this regime gives $\mathcal{O}((\log n)^{-1})$ rate of convergence) to deduce that

Theorem 6.2. *Assume as in Proposition 6.1. Let* $v \in C^\infty((0,\infty)\times\mathbb{R}^{\mathrm{d}};[0,1])\cap C([0,\infty)\times\mathbb{R}^{\mathrm{d}};[0,1])$ *solve the heat equation*

$$\begin{cases}\partial_t v = \frac{1}{2}\Delta v,\\ v(0,x) = p_0(x).\end{cases}$$

Then for all $t \in [0,T_n]$ *and* $x \in \mathbb{R}^d$,

$$\left|\mathbb{E}_{p_0}\left[w_t^n(x)\right] - v(\sigma^2 t, x)\right| = \mathcal{O}((\log n)^{-\min\{\alpha,1\}} + (\log n)^{T\lambda+\alpha}\cdot\omega(4n^{-\beta/6} b\log\log n)).$$

Remark 6.3. The bound in the new version of Proposition 3.2 was easily attained. By being more careful, it is possible to get a better, more technical result: pick $\gamma \in (0,1)$, and demand that

$$\begin{aligned} b_n &= b(\log n)^\gamma,\\ T_n &= T(\log n)^\gamma,\\ k_n &\leqslant \exp\bigl(k(\log n)^{1-\gamma}\bigr),\\ T\lambda &< b,\\ b + b\log\left(\frac{eT\lambda}{b}\right) + T\lambda &< 0,\\ \alpha &< 1-4\beta,\\ bk &< 1-4\beta-\alpha. \end{aligned}$$

Then the approximation to branching Brownian motion occurs at rate $\exp(-\mathcal{O}((\log n)^\gamma))$, meaning that the overall convergence occurs at rate

$$\exp(-\mathcal{O}((\log n)^\gamma)) + \exp((\lambda T+\alpha)(\log n)^\gamma)\cdot\omega(4n^{-\beta/6}b_n).$$

6.2 Constant η, large k

Choosing a scaling limit for this regime is a little more complicated. As we are letting k grow large, but insisting that we fix η, we see that we have no choice but to allow λ to grow large as well. We must be careful that it does not grow so rapidly that, for example, our expected number of lineages blows up.

There are two key relative scalings that we have to preserve. The first is

$$k_n T_n = o(b_n),$$

which is necessary to get that the probability of there being many expansive events is small. The same relation is also necessary for the dual to approximate branching Brownian motion. The second relation is

$$k_n^{b_n}\cdot n^{4\beta-1}\cdot T_n = o(1),$$

so that the probability of there being two simultaneously marked individuals is small.

So let $\mathrm{d}, \beta, u, s, R$ be as in Section 3. For $n \in \mathbb{N}$, let (k_n) be a sequence on $\{2,3,4,\ldots\}$ tending to infinity such that $k_n = \mathcal{O}(\log\log n)$. In particular pick k such that $k_n \leqslant k\log\log n$. Now for $n \in \mathbb{N}$ let $(u_n), (s_n)$ and (R_n) be as in Section 3.

Remark 6.4. Continuing on from Remarks 2.2 and 5.10, we once again expect that our results should hold for random k. Here it would suffice to consider k_n as in Remark 5.10, with the additional condition that $l_n = \mathcal{O}(\log\log n)$ and $N_n = \mathcal{O}(\log\log n)$.

Let Π^n be a Poisson point process on $[0,\infty)\times\mathbb{R}^{\mathrm{d}}$ with intensity measure $n\mathrm{d}t \otimes n^{\beta}\mathrm{d}x$ as in Section 3. For each $n \in \mathbb{N}$, let w^n be an SLFVE and Ξ^n be an SLFVE dual, in each case driven by Π^n, with parameters (u_n, s_n, R_n, k_n).

Let m_n be the jump intensity of a single lineage in Ξ^n. It is the same as in Section 3, so the total rate of jumps is once again $un^{2\beta}R^{\mathrm{d}}V_1$. Now $1 - s_n$ of the jumps will be from neutral events, and s_n of the jumps will be from expansive events. Thus the lineage is affected by expansive events at rate η, where

$$\eta = un^{2\beta}R^{\mathrm{d}}V_1 s_n = usR^{\mathrm{d}}V_1$$

which is independent of n. Let

$$\lambda_n = \eta k_n$$

which is now no longer independent of n. Note that $\lambda_n \geqslant \eta$.

Let σ^2 be as in Section 3.

We now have only one technical requirement. Let $\alpha > 0$, and let $b > 0$ be large enough or $T > 0$ small enough that

$$b\log\left(\frac{eTk\eta}{b}\right) + \alpha + kT\eta < 0. \tag{6.3}$$

Finally let

$$T_n = T\log\log n \qquad \text{and} \qquad b_n = b(\log\log n)^2.$$

As before we move on to adjusting our analysis of our previous argument.

(i) The maximal number of expansive events, as in Lemma 3.1, is now

$$\mathbb{P}\left[\mathrm{Expan}_{T_n}(\Xi^n) > b_n\right] = o((\log n)^{-(\alpha+T\eta k)\log\log n})$$

To see this, let $Z_n \sim \mathrm{Poisson}(T_n\eta)$. Now $b_n = \mathcal{O}((\log\log n)^2)$ and $T_n\eta = \mathcal{O}(\log\log n)$ so we may (eventually) apply the Chernoff bound to give that

$$\mathbb{P}[Z_n > b_n] \leqslant \left(\frac{eT_n\eta}{b_n}\right)^{b_n},$$

thus bounding the maximal number of expansive events by

$$k_n^{b_n}\cdot\mathbb{P}[Z_n > b_n] = \left(\frac{eT_nk_n\eta}{b_n}\right)^{b_n} \leqslant \left(\frac{eTk\eta}{b}\right)^{b_n} = o((\log n)^{-(\alpha+T\eta k)\log\log n})$$

by equation (6.3).

(ii) The probability of having simultaneously marked individuals, as in Proposition 3.2, is now

$$\mathbb{P}\left[\mathrm{Simul}_{T_n}(\Xi^n)\right] = o((\log n)^{-\alpha\log\log n})$$

To see this, it suffices to show that

$$k_n^{2b_n}\cdot n^{4\beta-1}\cdot\log\log n = o((\log n)^{-\alpha\log\log n})$$

which follows by recalling that $4\beta - 1 < 0$.

(iii) The expected number of lineages, conditioned on $\mathrm{NoSimul}_{T_n}(\Xi^n)$, as in Lemma 3.3, is now bounded by

$$\mathbb{E}\left[\mathrm{Lin}_{T_n}(\Xi^n) \middle| \mathrm{NoSimul}_{T_n}(\Xi^n)\right] = \exp(\eta(k_n - 1)T_n) \leqslant (\log n)^{T\eta k \log\log n}.$$

This also bounds the expected number of individuals, as in Corollary 3.4.

(iv) Applying these changes to Proposition 3.6, we find that the approximation to branching Brownian motion now occurs at rate $o((\log n)^{-\alpha \log\log n})$.

(v) Adjusting the definition of D_2 in Proposition 3.8 to $\{N_t^n \leqslant (\log n)^{(\alpha + T\eta k)\log\log n}\}$ gives our overall convergence at rate

$$\mathcal{O}((\log n)^{-\alpha \log\log n} + (\log n)^{(\alpha+T\eta k)\log\log n} \cdot \omega(4n^{-\beta/6}b_n)).$$

And so we can now bring these results together, along with Propostion 5.8, to deduce:

Theorem 6.5. *Let $\mathbb{d} \in \mathbb{N}$. Let $\beta \in (0, 1/4)$. Let $u \in (0,1)$, $s \in (0,1)$, $R > 0$ and $T > 0$. Let V_1 be the volume of the ball $B(0,R) \subseteq \mathbb{R}^{\mathbb{d}}$. Let $\eta = usR^{\mathbb{d}}V_1$. For $n \in \mathbb{N}$, let (k_n) be a sequence on $\{2, 3, 4, \ldots\}$ tending to infinity such that $k_n = \mathcal{O}(\log\log n)$.*

For each $n \in \mathbb{N}$, let

$$u_n = \frac{u}{n^{1-2\beta}} \qquad \text{and} \qquad s_n = \frac{s}{n^{2\beta}} \qquad \text{and} \qquad R_n = \frac{R}{n^\beta}.$$

Also let

$$\sigma^2 = \frac{uR^{\mathbb{d}}V_1}{\mathbb{d}}.$$

Let Π^n be a Poisson point process on $[0,\infty) \times \mathbb{R}^{\mathbb{d}}$ with intensity measure $n\mathrm{d}t \otimes n^\beta \mathrm{d}x$. Let $p_0 \in C_c(\mathbb{R}^{\mathbb{d}}; [0,1])$ be Hölder continuous and nontrivial.

For each $n \in \mathbb{N}$, let w^n be an SLFVE driven by Π^n with parameters (u_n, s_n, R_n, k_n). Then there exists $v \in C^\infty((0,\infty) \times \mathbb{R}^{\mathbb{d}}; [0,1]) \cap C([0,\infty) \times \mathbb{R}^{\mathbb{d}}; [0,1])$ solving the equation

$$\begin{cases} \partial_t v = \frac{1}{2}\Delta v + \eta(1-v), \\ v(0,x) = p_0(x). \end{cases}$$

such that for all $t \in [0, T\log\log n]$ and $x \in \mathbb{R}^{\mathbb{d}}$,

$$\left|\mathbb{E}_{p_0}\left[w_t^n(x)\right] - v(\sigma^2 t, x)\right| \to 0$$

uniformly in t and x as $n \to \infty$.

Furthermore we recall Remark 5.9 which gives another description of this v.

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